STRUCTURE OF THE STRING LINK CONCORDANCE GROUP AND HIRZEBRUCH-TYPE INVARIANTS

JAE CHOON CHA

ABSTRACT. We employ Hirzebruch-type invariants obtained from iterated p-covers to investigate concordance of links and string links. We show that the invariants naturally give various group homomorphisms of the string link concordance group into L-groups over number fields. We also obtain homomorphisms of successive quotients of the Cochran-Orr-Teichner filtration. As an application we show that the kernel of Harvey's ρ_n-invariant is large enough to contain a subgroup with infinite rank abelianization, modulo local knots. As another application, we show that recently discovered nontrivial 2-torsion examples of iterated Bing doubles lying at an arbitrary depth of the Cochran-Orr-Teichner filtration are independent over $\mathbb{Z}_2$ as links, in an appropriate sense. We also construct similar examples of infinite order links which are independent over $\mathbb{Z}$.

1. INTRODUCTION AND MAIN RESULTS

In [2], the author defined new Hirzebruch-type intersection form defect invariants from towers of iterated p-covers, and gave applications to link concordance and homology cobordism. The aim of this paper is to employ the invariants to reveal further information on the structure of the string link concordance group and the set of link concordance classes. In this paper manifolds are assumed to be topological and oriented, and submanifolds are locally flat. All results hold in the smooth category as well.

1.1. **Hirzebruch-type invariants.** We start with a quick review of the invariants defined in [2]. Throughout this paper, p denotes a prime. For a CW-complex X, a tower

$$X_n \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0 = X$$

consisting of connected covers X_i of X is called a *p-tower of height n for X* if each $X_i \to X_{i-1}$ is a regular cover whose covering transformation group is a finite abelian p-group. Suppose M is a closed 3-manifold. For a p-tower $\{M_i\}$ of height n and a character $\phi\colon \pi_1(M_n) \to \mathbb{Z}_d$ with d a power of p such that $(M_n, \phi) = 0$ in the topological bordism group $\Omega_3^{top}(B\mathbb{Z}_d)$, an invariant

$$\lambda(M_n, \phi) \in L^0(\mathbb{Q}(\zeta_d))$$

is defined, where $\zeta_d = \exp(2\pi\sqrt{-1}/d)$ is the dth primitive root of unity and $L^0(\mathbb{Q}(\zeta_d))$ is the group of Witt classes of nonsingular hermitian forms over $\mathbb{Q}(\zeta_d)$. (As usual, $\mathbb{Q}(\zeta_d)$ is endowed with the involution $\bar{\zeta}_d = \zeta_d^{-1}$.) Basically $\lambda(M_n, \phi)$ is the difference of the Witt class of the $\mathbb{Q}(\zeta_d)$-coefficient intersection form of a 4-manifold W bounded by M_n over $\mathbb{Z}_d$ and that of the untwisted intersection form of W. (For more details the reader is referred to [2].)

We remark that in order to define the invariant it suffices to specify ϕ as a map of a subgroup of $\pi_1(M)$, without considering the p-tower $\{M_i\}$: we call $\phi\colon H \to \mathbb{Z}_d$

a ($\mathbb{Z}_d$-valued) *p-virtual character* of a group G if ϕ is a group homomorphism of a subgroup H in G such that the index $[G : H]$ is finite and a power of p. For any p-virtual character ϕ of $\pi_1(M)$, it can be seen by an easy induction that there is a p-tower $\{M_i\}$ of M whose highest term M_n is the cover M_ϕ determined by H, i.e., the image of $\pi_1(M_\phi)$ under the map induced by the covering projection is exactly $H \subset \pi_1(M)$. The invariant $\lambda(M_\phi, \phi)$ is independent of the choice of $\{M_i\}$ (provided it is defined). Since in this paper we always define a p-virtual character by constructing a p-tower, we regard a p-virtual character of $\pi_1(M)$ as endowed with a p-tower of M. In other words, we think of a pair $(\{M_i\}, \phi)$ of a p-tower $\{M_i\}$ and $\phi\colon \pi_1(M_n) \to \mathbb{Z}_d$. We call such a pair a $\mathbb{Z}_d$-*valued p-structure of height* n for M. (The order d is always assumed to be a power of p.)

1.2. **Invariants of the string link concordance group.** In [2], it was shown that the Hirzebruch-type invariant defined from a p-structure of the surgery manifold of a link in S^3 gives an obstruction to being a slice link. In this paper we also think of *string links* in the sense of Habegger and Lin [12, 13], which has the advantage that there is a well-defined group structure on the set of concordance classes; we denote the group of concordance classes of m-component string links by $\mathcal{C}^{\mathrm{SL}}(m)$, or simply $\mathcal{C}^{\mathrm{SL}}$.

The invariants in [2] give rise to invariants of $\mathcal{C}^{\mathrm{SL}}$ with values in $L^0(\mathbb{Q}(\zeta_d))$ via the closures of string links. As the first main result of this paper, we show that the invariants restricted to a large class of string links become *group homomorphisms*; we call a string link β an $\widehat{F}$-*string link* if the closure of β is an $\widehat{F}$-link in the sense of Levine [15]. Roughly speaking, it means that the fundamental group of the complement of the closure of β has Levine's algebraic closure [15] identical with that of a free group (and the preferred longitudes are trivial in the algebraic closure). We note that $\widehat{F}$-(string) links form the largest known class of (string) links with vanishing Milnor's $\bar{\mu}$-invariant; it is a big open problem in link theory whether all (string) links with vanishing Milnor's $\bar{\mu}$-invariants are $\widehat{F}$-(string) links.

Let $\widehat{\mathcal{C}}^{\mathrm{SL}} = \widehat{\mathcal{C}}^{\mathrm{SL}}(m)$ be the subgroup of $\mathcal{C}^{\mathrm{SL}}$ consisting of (the classes of) $\widehat{F}$-string links. An advantage of $\widehat{F}$-links is that p-structures of the surgery manifolds are naturally described in terms of those of a fixed space $X = \bigvee^m S^1$, the wedge of m circles. Namely, in Section 2, we show that every p-structure $\mathcal{T} = (\{X_i\}, \theta)$ of height n for X *canonically* determines a p-structure $(\{M_i\}, \phi)$ of height n for the surgery manifold M of the closure of an $\widehat{F}$-string link β, and any p-structure for M arises in this way. We define $\lambda_{\mathcal{T}}(\beta) \in L^0(\mathbb{Q}(\zeta_d))$ to be the Hirzebruch-type invariant $\lambda(M_n, \phi)$. The result stated below essentially says that $\lambda_{\mathcal{T}}(\beta)$ is additive under product of $\widehat{F}$-string links.

Theorem 1.1. *For any $\mathbb{Z}_d$-valued p-structure $\mathcal{T} = (\{X_i\}, \theta)$ for X, the invariant $\lambda_{\mathcal{T}}(\beta)$ is well-defined for any $\widehat{F}$-string link, and this gives rise to a group homomorphism*

$$\lambda_{\mathcal{T}}\colon \widehat{\mathcal{C}}^{\mathrm{SL}} \longrightarrow L^0(\mathbb{Q}(\zeta_d)).$$

We remark that $\mathcal{C}^{\mathrm{SL}}$ and $\widehat{\mathcal{C}}^{\mathrm{SL}}$ contain the knot concordance group as a summand; $\mathcal{C}^{\mathrm{SL}}(1) = \widehat{\mathcal{C}}^{\mathrm{SL}}(1)$ is isomorphic to the knot concordance group and there is an obvious split injection $\bigoplus^m \mathcal{C}^{\mathrm{SL}}(1) \to \widehat{\mathcal{C}}^{\mathrm{SL}}(m) \subset \mathcal{C}^{\mathrm{SL}}(m)$ whose image is the (normal) subgroup generated by "local knots". (A precise definition of a local knot is given in Section 4.) In order to study the sophistication peculiar to links, modulo knot concordance, we consider the quotient of the string link concordance group modulo local knots. A subclass of our invariants can be used to investigate this. Let x_i be the loop in $X = \bigvee^m S^1$ representing the ith circle. We say that a p-structure

$\mathcal{T} = (\{X_i\}, \theta)$ of height n for X is *locally trivial* if any element in $\pi_1(X_n)$ which projects to a conjugate of a power of $[x_i]$ in $\pi_1(X)$ is in the kernel of θ.

Addendum to Theorem 1.1. *If $\mathcal{T}$ is locally trivial, then $\lambda_{\mathcal{T}}$ induces a group homomorphism*

$$\lambda_{\mathcal{T}} : \frac{\widehat{\mathcal{C}}^{\mathrm{SL}}}{\langle local\ knots \rangle} \longrightarrow L^0(\mathbb{Q}(\zeta_d))$$

where $\langle local\ knots \rangle$ denotes the subgroup generated by local knots.

In fact our results on $\widehat{F}$-string links also hold for a (potentially) larger class of string links, namely $\mathbb{Z}_{(p)}$-*coefficient* $\widehat{F}$-*string links* in the sense of [2], where $\mathbb{Z}_{(p)}$ designates the localization of $\mathbb{Z}$ away from p; the definition of a $\mathbb{Z}_{(p)}$-coefficient $\widehat{F}$-string link is identical with that of a $\widehat{F}$-string link except that the algebraic closure with respect to $\mathbb{Z}_{(p)}$-coefficients (see [1]) is used in place of Levine's algebraic closure. Since there is a natural transformation from Levine's algebraic closure to the algebraic closure with $\mathbb{Z}_{(p)}$-coefficients, an $\widehat{F}$-(string) link is a $\mathbb{Z}_{(p)}$-coefficient $\widehat{F}$-(string) link. In all results in this paper, the term "$\widehat{F}$-(string) link" can be understood as this $\mathbb{Z}_{(p)}$-coefficient analogue.

1.3. **Structure of Cochran-Orr-Teichner filtration.** We use the homomorphism $\lambda_{\mathcal{T}}$ to investigate the structure of the Cochran-Orr-Teichner filtration

$$\cdots \subset \mathcal{F}^{\mathrm{SL}}_{(n.5)} \subset \mathcal{F}^{\mathrm{SL}}_{(n)} \subset \cdots \subset \mathcal{F}^{\mathrm{SL}}_{(1)} \subset \mathcal{F}^{\mathrm{SL}}_{(0.5)} \subset \mathcal{F}^{\mathrm{SL}}_{(0)} \subset \mathcal{C}^{\mathrm{SL}}$$

of the string link concordance group [10, 14]. Here $\mathcal{F}^{\mathrm{SL}}_{(h)}$ ($h \in \frac{1}{2}\mathbb{Z}_{\geq 0}$) denotes the subgroup (of concordance classes) of string links whose closures are (h)-solvable in the sense of [10]. Harvey proved an interesting result on this filtration that $\mathcal{F}^{\mathrm{SL}}_{(n)}/\mathcal{F}^{\mathrm{SL}}_{(n.5)}$ is highly nontrivial [14]. More precisely, she considered the subgroup $\mathcal{B}(m)$ of (concordance classes of) boundary string links in $\mathcal{C}^{\mathrm{SL}}$, and the induced filtration $\{\mathcal{BF}^{\mathrm{SL}}_{(h)} = \mathcal{F}^{\mathrm{SL}}_{(h)} \cap \mathcal{B}(m)\}$ of $\mathcal{B}(m)$. She defined a real-valued group homomorphism

$$\rho_n : \frac{\mathcal{BF}^{\mathrm{SL}}_{(n)}}{\mathcal{BF}^{\mathrm{SL}}_{(n.5)} \cdot \langle \text{local knots} \rangle} \longrightarrow \mathbb{R},$$

and using it, proved that the abelianization of $\mathcal{BF}^{\mathrm{SL}}_{(n)}/(\mathcal{BF}^{\mathrm{SL}}_{(n.5)} \cdot \langle \text{local knots} \rangle)$ has infinite rank for any $m > 1$ and n. (In the original statement in [14], $\mathcal{BF}^{\mathrm{SL}}_{(n+1)}$ was used in place of $\mathcal{BF}^{\mathrm{SL}}_{(n.5)}$; it was generalized to the above form in a subsequent work of Cochran and Harvey [7].)

Our invariant reveals further information on the filtration. Let $\widehat{\mathcal{F}}^{\mathrm{SL}}_{(h)} = \mathcal{F}^{\mathrm{SL}}_{(h)} \cap \widehat{\mathcal{C}}^{\mathrm{SL}}$. We remark that $\mathcal{BF}^{\mathrm{SL}}_{(h)}$ is a subgroup of $\widehat{\mathcal{F}}^{\mathrm{SL}}_{(h)}$ since a boundary string link is an $\widehat{F}$-string link. It is known that there are $\widehat{F}$-(string) links which are not concordant to any boundary (string) link [9]; the examples in [9] illustrate that $\mathcal{BF}^{\mathrm{SL}}_{(1)}$ is a proper subgroup of $\widehat{\mathcal{F}}^{\mathrm{SL}}_{(1)}$ for $m > 1$.

The following result may be viewed as a refinement of Theorem 1.1:

Theorem 1.2. *For any p-structure $\mathcal{T} = (\{X_i\}, \theta)$ of height n, $\lambda_{\mathcal{T}}$ gives rise to a homomorphism*

$$\widehat{\mathcal{F}}^{\mathrm{SL}}_{(n)}/\widehat{\mathcal{F}}^{\mathrm{SL}}_{(n.5)} \subset \widehat{\mathcal{C}}^{\mathrm{SL}}/\widehat{\mathcal{F}}^{\mathrm{SL}}_{(n.5)} \longrightarrow L^0(\mathbb{Q}(\zeta_d)).$$

In addition, if $\mathcal{T}$ is locally trivial, then $\lambda_{\mathcal{T}}$ induces a homomorphism

$$\frac{\widehat{\mathcal{F}}^{\mathrm{SL}}_{(n)}}{\widehat{\mathcal{F}}^{\mathrm{SL}}_{(n.5)} \cdot \langle local\ knots \rangle} \subset \frac{\widehat{\mathcal{C}}^{\mathrm{SL}}}{\widehat{\mathcal{F}}^{\mathrm{SL}}_{(n.5)} \cdot \langle local\ knots \rangle} \longrightarrow L^0(\mathbb{Q}(\zeta_d)).$$

Using this theorem we show that for any $m > 1$ and n, the abelianization of $\widehat{\mathcal{F}}^{\mathrm{SL}}_{(n)}/(\widehat{\mathcal{F}}^{\mathrm{SL}}_{(n.5)} \cdot \langle \text{local knots} \rangle)$ is of infinite rank (see Theorem 5.4). As an immediate corollary of (the proof of) this, we obtain an alternative proof of Harvey's result that the abelianization of $\mathcal{BF}^{\mathrm{SL}}_{(n)}/(\mathcal{BF}^{\mathrm{SL}}_{(n.5)} \cdot \langle \text{local knots} \rangle)$ has infinite rank. Moreover, we prove that the kernel of Harvey's homomorphism ρ_n is large:

Theorem 1.3. *For any $m > 1$ and n, the abelianization of the kernel of ρ_n on $\mathcal{BF}^{\mathrm{SL}}_{(n)}/(\mathcal{BF}^{\mathrm{SL}}_{(n.5)} \cdot \langle \text{local knots} \rangle)$ has infinite rank.*

In the proof of Theorem 1.3 we construct concrete examples of (boundary) string links which have vanishing ρ_n-invariants but are linearly independent (over $\mathbb{Z}$) in the abelianization. We remark that recently Cochran, Harvey, and Leidy [8] have announced a (different) proof of the following statement, which is an immediate consequence of Theorem 1.3: viewing ρ_n as a homomorphism on $\mathcal{BF}^{\mathrm{SL}}_{(n)}/\mathcal{BF}^{\mathrm{SL}}_{(n.5)}$, its kernel is infinitely generated (without taking the abelianization and without taking the quotient by local knots).

As a part of the proof Theorem 1.2, we investigate the relationship between the solvability of a 3-manifold and its (iterated) covers. A related result we call *Covering Solution Theorem* seems interesting by its own. Roughly speaking, it says: *taking an abelian p-cover, the solvability decreases by one.* (We need some technical assumptions; for precise statements, refer to Definition 3.1 and Theorem 3.5.) Further applications of Covering Solution Theorem will be discussed in later papers.

1.4. **Disk basings and independence of links.** We also investigate the structure of the Cochran-Orr-Teichner filtration of (spherical) links. Let $\mathcal{C}^{\mathrm{L}}(m)$ be the set of concordance classes of m-component links in S^3, and let

$$\cdots \subset \mathcal{F}^{\mathrm{L}}_{(n.5)} \subset \mathcal{F}^{\mathrm{L}}_{(n)} \subset \cdots \subset \mathcal{F}^{\mathrm{L}}_{(1)} \subset \mathcal{F}^{\mathrm{L}}_{(0.5)} \subset \mathcal{F}^{\mathrm{L}}_{(0)} \subset \mathcal{C}^{\mathrm{L}}(m)$$

be the Cochran-Orr-Teichner filtration of $\mathcal{C}^{\mathrm{L}}(m)$, namely, $\mathcal{F}^{\mathrm{L}}_{(h)}$ is the collection (of concordance classes) of (h)-solvable links in the sense of [10].

In contrast to $\mathcal{C}^{\mathrm{SL}}(m)$, $\mathcal{C}^{\mathrm{L}}(m)$ does not have a natural group structure for $m > 1$. One can think of a connected sum of two links, but in general it does not give a unique (concordance class of a) link. To define a connected sum one may pass through string links; by choosing a "disk basing" of a link L in the sense of [12], one obtains a string link β whose closure is L. Here a disk basing is an embedded 2-disk in S^3 which meets components of L transversely at prescribed positive intersection points, and β is obtained by cutting S^3 along the 2-disk. Given two links L_1 and L_2, a connected sum of L_1 and L_2 is defined to be the closure of the product of two string links obtained from L_1 and L_2 by choosing disk basings. Similarly we define a connected sum of finitely many links by choosing disk basings and an order of the links.

Regarding the role of disk basings, we think of a strong notion of "independence" of links as below. For a link L and $a \in \mathbb{Z}$, we denote by aL a copies of L if $a \geq 0$, and $|a|$ copies of $-L$ for $a < 0$, where $-L$ designates the inverse of L. A family $\{L_i\}_{i \in I}$ of links indexed by a set I is called *independent over $\mathbb{Z}$ modulo $\mathcal{F}^{\mathrm{L}}_{(h)}$ and local knots* if the following holds: for any sequence $\{a_i\}_{i \in I}$ of integers which are all zero but finitely many, if a connected sum of the $a_i L_i$ $(i \in I)$ with some local knots added is (h)-solvable, then $a_i = 0$ for all $i \in I$. Roughly, this means that any connected sum of copies of a link in $\{L_i\}_{i \in I}$ is never obtained as a connected sum of copies of other links in $\{L_i\}_{i \in I}$ even modulo (h)-solvability and local knots.

Theorem 1.4. *For any $m > 1$ and n, there are infinitely many m-component links which are in $\mathcal{F}^{\mathrm{L}}_{(n)}$, have vanishing ρ_n-invariant, and are independent over $\mathbb{Z}$ modulo $\mathcal{F}^{\mathrm{L}}_{(n.5)}$ and local knots.*

In [2], it was shown that the nth iterated Bing doubles of some knots are in $\mathcal{F}^{\mathrm{L}}_{(n)}$ and have vanishing ρ_n-invariant but are not in $\mathcal{F}^{\mathrm{L}}_{(n+1)}$. Using the techniques used in the proof Theorem 1.4, we generalize this result by showing that the knots can be chosen in such a way that their nth iterated Bing doubles satisfy the conclusion of Theorem 1.4.

In [2], it was also shown that there are 2-torsion links in an arbitrary depth of the Cochran-Orr-Teichner filtration; the examples are the nth iterated Bing doubles of certain (negatively) amphichiral knots. (We call a link L *2-torsion* if a connected sum of L and L itself is a slice link; we remark that the ρ_n-invariants vanish for any 2-torsion link.) We consider the following analogous notion of independence for 2-torsion links: a family $\{L_i\}_{i\in I}$ of 2-torsion links is said to be *independent over $\mathbb{Z}_2$ modulo $\mathcal{F}^{\mathrm{L}}_{(h)}$ and local knots* if a subset J of I is an empty set whenever a connected sum of $\{L_j\}_{j\in J}$ with some local knots added is (h)-solvable. We show that the 2-torsion examples of iterated Bing doubles in [2] can be chosen in such a way that they are independent in this sense:

Theorem 1.5. *For any n, there are infinitely many amphichiral knots whose nth iterated Bing doubles are 2-torsion, in $\mathcal{F}^{\mathrm{L}}_{(n)}$, and independent over $\mathbb{Z}_2$ modulo $\mathcal{F}^{\mathrm{L}}_{(n+1.5)}$ and local knots.*

The proofs of Theorems 1.4 and 1.5 involve a careful analysis of the effect of a disk basing change on p-structures, utilizing results of Habegger and Lin [12, 13]. See Section 6 for details.

2. Invariants of string links

We review basic definitions and fix notations for string links. Fix m distinct points $p_1, \ldots, p_m$ in the interior of D^2. An *m-component string link* (or *m-string link*) is defined to be the image of a locally flat proper embedding of $\{1, \ldots, m\} \times [0,1]$ into $D^2 \times [0,1]$ sending (i,t) to (p_i, t) for $i = 1, \ldots, m$ and $t = 0, 1$. The product of two string links is defined by juxtaposition. The concordance classes of string links (in the sense of [13]) form a group under the product operation; the inverse $-\beta$ is the mirror image of β (about $D^2 \times \{\frac{1}{2}\}$) with reversed orientation. We denote this group by $\mathcal{C}^{\mathrm{SL}} = \mathcal{C}^{\mathrm{SL}}(m)$ as in the introduction.

Gluing $D^2 \times \{0\}$ and $D^2 \times \{1\}$ along the identity map of D^2 and then filling it in with a solid torus in such a way that the image of $\{*\} \times [0,1]$ bounds a disk for $* \in \partial D^2$, we obtain from β a link L in S^3, which is called the *closure* of β. It is known that the closure L is a slice link if and only if β is concordant to a trivial string link [13]. We denote the exterior of β and L by E_β and E_L, respectively. Also, the surgery manifold of L is denoted by M_β or M_L. There are natural inclusions $E_\beta \to E_L \to M_\beta$.

As in the introduction, let $X = \bigvee^m S^1$, the wedge of m circles. Fix an embedding $f \colon X \to D^2 - \{p_1, \ldots, p_m\}$ such that the wedge point $* \in X$ is sent to a fixed basepoint on ∂D^2 and $x \to (f(x), 0)$ defines a map $\mu \colon X \to E_\beta$ which sends the ith circle of X to a positive meridian of the ith component of β. We call μ the *preferred meridian map* of β. Sometimes the composition

$$X \xrightarrow{\mu} E_\beta \longrightarrow M_\beta$$

is also referred to as the preferred meridian map.

We call a string link β an *$\widehat{F}$-string link* if its closure L is an $\widehat{F}$-link in the sense of Levine [15], that is, the composition

$$X \xrightarrow{\mu} E_\beta \longrightarrow E_L$$

induces an isomorphism $\widehat{\pi_1(X)} \to \widehat{\pi_1(E_L)}$ and the preferred longitudes of L are in the kernel of the natural map $\pi_1(E_L) \to \widehat{\pi_1(E_L)}$. Here $\widehat{G}$ denotes the algebraic closure of a group G defined by Levine [15].

As a (potentially) generalized notion, we say that β is a $\mathbb{Z}_{(p)}$-*coefficient* $\widehat{F}$-*string link* if β satisfies the defining condition of an $\widehat{F}$-string link with Levine's algebraic closure replaced by the "algebraic closure with respect to $\mathbb{Z}_{(p)}$-coefficients" defined in [1]. Equivalently, β is a $\mathbb{Z}_{(p)}$-*coefficient* $\widehat{F}$-*string link* if its closure is a $\mathbb{Z}_{(p)}$-*coefficient* $\widehat{F}$-*link* in the sense of [2]. Since an $\widehat{F}$-string link is a $\mathbb{Z}_{(p)}$-coefficient $\widehat{F}$-string link for any p and since all results in this paper hold for the latter as well as the former, from now on $\widehat{G}$ denotes the algebraic closure of G with respect to $\mathbb{Z}_{(p)}$-coefficients, and an "$\widehat{F}$-string link" designates a $\mathbb{Z}_{(p)}$-coefficient $\widehat{F}$-string link, as an abuse of terminology. It can be seen that the set $\widehat{\mathcal{C}}^{\mathrm{SL}}$ of classes of $\widehat{F}$-string links is closed under the group operations of $\mathcal{C}^{\mathrm{SL}}$, that is, $\widehat{\mathcal{C}}^{\mathrm{SL}}$ is a subgroup in $\mathcal{C}^{\mathrm{SL}}$.

Suppose $Y \to Z$ is a map between CW-complexes. Then a $\mathbb{Z}_d$-valued p-structure of height n for Z induces a $\mathbb{Z}_d$-valued p-structure of height n for Y via pullback along $Y \to Z$. We recall from [2] that $Y \to Z$ is called a p-*tower map* if pullback gives rise to a 1-1 correspondence between $\mathbb{Z}_d$-valued p-structures of height n for Y and Z for any n and d. For a more precise description, see Definition 3.4 of [2]. The following was proved in [2]:

Lemma 2.1 (Proposition 6.3 of [2]). *For any $\widehat{F}$-string link β, any meridian map $X \to M_\beta$ is a p-tower map. In particular, the preferred meridian map into M_β is a p-tower map.*

Suppose $\mathcal{T} = (\{X_k\}, \theta)$ is a $\mathbb{Z}_d$-valued p-structure of height n for X. Since the preferred meridian map $X \to M_\beta$ of β is a p-tower map, there is a uniquely determined $\mathbb{Z}_d$-valued p-structure $(\{M_k\}, \phi)$ of height n for M_β which induces $\mathcal{T}$ via pullback. In [2, Definition 2.2], an invariant $\lambda(M_n, \phi) \in L^0(\mathbb{Q}(\zeta_d))$ is defined when $(M_n, \phi) = 0$ in the topological bordism group $\Omega_3^{top}(B\mathbb{Z}_d)$.

Lemma 2.2. *If β is an $\widehat{F}$-string link, then $(M_n, \phi) = 0$ in $\Omega_3^{top}(B\mathbb{Z}_d)$.*

The proof of Lemma 2.2 is postponed. By Lemma 2.2, the following definition is meaningful for any $\widehat{F}$-string link:

Definition 2.3. For a p-structure $\mathcal{T} = (\{X_k\}, \theta)$ for X and an $\widehat{F}$-string link β, we define

$$\lambda_{\mathcal{T}}(\beta) = \lambda(M_n, \phi) \in L^0(\mathbb{Q}(\zeta_d))$$

where $(\{M_k\}, \phi)$ is the p-structure for M_β induced by $\mathcal{T}$ as above.

The main aim of this section is to prove the following additivity:

Theorem 2.4. *If β and β' are $\widehat{F}$-string links, then for any p-structure $\mathcal{T}$, we have*

$$\lambda_{\mathcal{T}}(\beta \cdot \beta') = \lambda_{\mathcal{T}}(\beta) + \lambda_{\mathcal{T}}(\beta') \quad in \ L^0(\mathbb{Q}(\zeta_d)).$$

By [2, Theorem 6.2], $\lambda_T(\beta) = 0$ if β is concordant to a trivial string link. Combining this with Theorem 2.4, we obtain (the first part of) Theorem 1.1 stated in the introduction: for any p-structure $\mathcal{T} = (\{X_k\}, \theta)$, the invariant $\lambda_{\mathcal{T}}(-)$ gives rise to a group homomorphism

$$\lambda_{\mathcal{T}} \colon \widehat{\mathcal{C}}^{\mathrm{SL}} \longrightarrow L^0(\mathbb{Q}(\zeta_d)).$$

The remaining part of this section is devoted to the proof of Lemma 2.2 and Theorem 2.4.

Proof of Lemma 2.2. Suppose β is an $\widehat{F}$-string link. Suppose $\{M_i\}$ is a p-tower of height n for M_β and $\phi\colon \pi_1(M_n) \to \mathbb{Z}_d$ is a character. Let $\{X_i\}$ be the p-tower for X and $\theta\colon \pi_1(X_n) \to \mathbb{Z}_d$ be the character which are induced via pullback along $X \to M_\beta$. Since $X \to M_\beta$ is a p-tower map, the lift $X_n \to M_n$ of $X \to M_\beta$ induces an isomorphism

$$\mathrm{Hom}(\pi_1(M_n), \mathbb{Z}_{p^r}) \cong \mathrm{Hom}(\pi_1(X_n), \mathbb{Z}_{p^r})$$

for any r. Since

$$H_1(-) \otimes \mathbb{Z}_{p^r} \cong \mathrm{Hom}(H_1(-), \mathbb{Z}_{p^r}) \cong \mathrm{Hom}(\pi_1(-), \mathbb{Z}_{p^r})$$

we have

$$H_1(M_n) \otimes \mathbb{Z}_{p^r} \cong H_1(X_n) \otimes \mathbb{Z}_{p^r}.$$

Since X_n is a 1-complex, $H_1(X_n) \otimes \mathbb{Z}_{p^r} = (\mathbb{Z}_{p^r})^s$ for all r, where $s = \beta_1(X_n)$ is the first Betti number. Looking at the case $r = 1$ and the case of a sufficiently large r, one can see that $H_1(M_n)$ is p-torsion free, that is, $H_1(M_n)$ has no nontrivial p-primary summand. From this it follows that $\phi\colon \pi_1(M_n) \to \mathbb{Z}_d$ factors through $H_1(M_n)/\mathrm{torsion}$, since d is a power of p. Furthermore the induced map $H_1(M_n)/\mathrm{torsion} \to \mathbb{Z}_d$ factors through the projection $\mathbb{Z} \to \mathbb{Z}_d$ since $H_1(M_n)/\mathrm{torsion}$ is a free abelian group.

From the above observation, it follows that $(M_n, \phi) \in \Omega_3^{top}(B\mathbb{Z}_d)$ is contained in the image of $\Omega_3^{top}(B\mathbb{Z}) \to \Omega_3^{top}(B\mathbb{Z}_d)$. Therefore, $(M_n, \phi) = 0$ in $\Omega_3^{top}(B\mathbb{Z}_d)$ since $\Omega_3^{top}(B\mathbb{Z}) = 0$ (e.g., by the Atiyah-Hirzebruch spectral sequence). □

Let $M = M_\beta$, $M' = M_{\beta'}$, and $N = M_{\beta \cdot \beta'}$ be the surgery manifolds of the closures of β, β', and $\beta \cdot \beta'$, respectively. In order to prove Theorem 2.4, as in [11, 14] we consider a "standard" cobordism W between $M \cup M'$ and N, which can be described as follows: attaching a 1-handle to $(M \cup M') \times [0, 1]$, we obtain a cobordism from $M \cup M'$ to $M\#M'$. And then attaching m 2-handles along the attaching spheres in $M\#M'$ as illustrated in Figure 1, we obtain a cobordism from $M\#M'$ to N. (The last two surgery diagrams in Figure 1 are equivalent by handle sliding and cancellation.) Our W is obtained by concatenating these cobordisms.
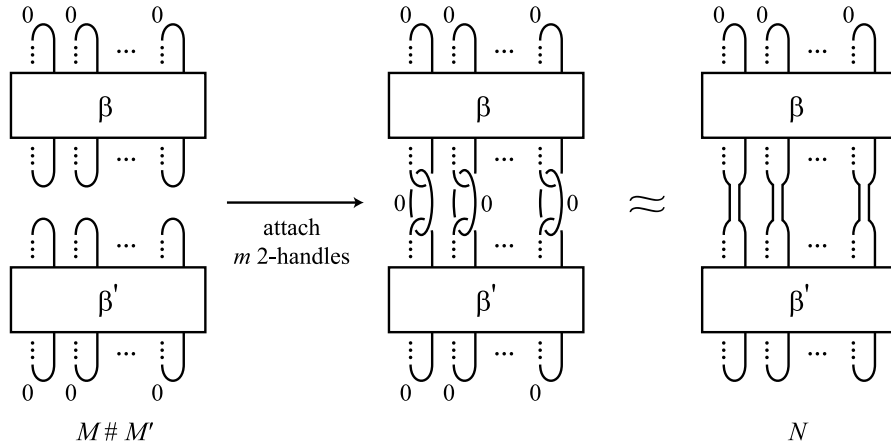

FIGURE 1. A cobordism from $M\#M'$ to N

Let $(\{M_k\}, \phi)$, $(\{M_k'\}, \phi')$ and $(\{N_k\}, \psi)$ be the p-structures of height n for M, M' and N, respectively, which are determined by a given p-structure $\mathcal{T} = (\{X_k\}, \theta)$ of height n for X. The first step of our proof is a construction of a bordism between (M_n, ϕ), (M_n', ϕ'), (N_n, ψ) using W.

Lemma 2.5. *There is a p-tower $\{W_k\}$ of W which induces $\{M_k\}$, $\{M'_k\}$, and $\{N_k\}$ by pullback along the inclusions. Furthermore, there is a character $\pi_1(W_n) \to \mathbb{Z}_d$ which restricts to ϕ, ϕ', and ψ on $\pi_1(M_n)$, $\pi_1(M'_n)$, and $\pi_1(N_n)$, respectively.*

In order to prove Lemma 2.5, we need the following facts which were proved in [15, 1, 2]: for CW-complexes or groups A and B, we say that $f\colon A \to B$ is *2-connected with respect to $\mathbb{Z}_{(p)}$-coefficients* if f induces an isomorphism on $H_1(-;\mathbb{Z}_{(p)})$ and an epimorphism on $H_2(-;\mathbb{Z}_{(p)})$. We remark that the defining condition holds for $\mathbb{Z}_{(p)}$-coefficients if and only if it holds for $\mathbb{Z}_p$-coefficients. In this section, for convenience we simply say that a map is 2-connected when it is 2-connected with respect to $\mathbb{Z}_{(p)}$ (or $\mathbb{Z}_p$)-coefficients.

Lemma 2.6. *In what follows A, B, and B_i are CW-complexes with finite 2-skeletons.*

  (0) *For any group G, the natural map $G \to \widehat{G}$ is 2-connected [15, 1].*
  (1) *If $f\colon A \to B$ is 2-connected, then f induces an isomorphism $\widehat{\pi_1(A)} \to \widehat{\pi_1(B)}$ [15, 1].*
  (2) *As a weak converse to (1), if $\{f_i\colon A \to B_i\}$ is a finite collection of maps inducing isomorphisms $\widehat{\pi_1(A)} \to \widehat{\pi_1(B_i)}$, then there is a 2-connected map $g\colon A \to Z = (a\ K(G,1)$-space with finite 2-skeleton$)$ such that for each i there is a 2-connected map $B_i \to Z$ making the diagram*

$$A \xrightarrow{\ f_i\ } B_i$$
$$g \searrow \qquad \swarrow$$
$$Z$$

commute [2, Proof of Proposition 3.9].

  (3) *Any $f\colon A \to B$ inducing an isomorphism $\widehat{\pi_1(A)} \to \widehat{\pi_1(B)}$ is a p-tower map [2, Proposition 3.9].*

We remark that although [2, Proof of Proposition 3.9] discusses a special case of Lemma 2.6 (2) that the family $\{f_i\}$ consists of only one map, exactly the same argument proves the above generalized case.

Proof of Lemma 2.5. Let $X' = X$ and $\mu\colon X \to E_\beta \to M$ and $\mu'\colon X' \to E_{\beta'} \to M'$ be the preferred meridian maps of β and β'. By Lemma 2.1, μ and μ' induce isomorphisms $\widehat{\pi_1(X)} \to \widehat{\pi_1(M)}$ and $\widehat{\pi_1(X')} \to \widehat{\pi_1(M')}$. By Lemma 2.6 (2), it follows that there are 2-connected maps of M, M' into a CW-complex Z with finite 2-skeleton making the following diagram commute:

Note that W has the homotopy type of

$$\bigl(M \vee M'\bigr) \cup \bigl(m\ 2\text{-disks attached along } \mu_*(x_i)\mu'_*(x'_i)^{-1}\bigr)$$

where x_i and x'_i denote the paths representing the ith circle in X and X', respectively. It follows that there is a map $W \to Z$ making the above diagram commute.

Let $\mu''\colon X'' \to N$ be the preferred meridian map of $\beta \cdot \beta'$, where $X'' = X$. It can be seen from Figure 1 that the preferred meridians of β, β', and $\beta \cdot \beta'$ are homotopic in W. Therefore, we have the following commutative diagram which extends the above one:

By Lemma 2.6 (3), the composition map $X \to Z$ is a p-tower map. So, given $\mathcal{T} = (\{X_k\}, \theta)$, there is a p-tower $\{Z_k\}$ for Z and a character of $\pi_1(Z_n)$ inducing $\{X_k\}$ and θ via pullback. Now, by their definition, $\{M_k\}$, $\{M_k'\}$, and $\{N_k\}$ are identical with the pullback p-towers induced by $\{Z_k\}$, and similarly for characters. It follows that the p-tower $\{W_k\}$ for W induced by $\{Z_k\}$ and the character of $\pi_1(W_n)$ induced by that of $\pi_1(Z_n)$ have the claimed properties. $\qquad\square$

Let $\{(M \cup M')_k\}$ and $\{(M \vee M')_k\}$ be the pullback p-towers of $M \cup M'$ and $M \vee M'$ determined by $\{Z_k\}$ in the above proof.

Lemma 2.7. $H_2((M \cup M')_k; \mathbb{Z}_{(p)}) \to H_2(W_k; \mathbb{Z}_{(p)})$ *is surjective for all k.*

We remark that our argument below also shows the conclusion for $\mathbb{Z}_p$-coefficients.

Proof. Since $(M \vee M')_k$ has the homotopy type of $(M \cup M')_k \cup$ (1-cells),

$$H_2((M \cup M')_k; \mathbb{Z}_{(p)}) \longrightarrow H_2((M \vee M')_k; \mathbb{Z}_{(p)})$$

is surjective by the long exact sequence of the pair $((M \vee M')_k, (M \cup M')_k)$. Therefore, it suffices to show that

$$H_2((M \vee M')_k; \mathbb{Z}_{(p)}) \longrightarrow H_2(W_k; \mathbb{Z}_{(p)})$$

is surjective. Let $Y = \bigvee^m S^1$, and let $Y \to X \vee X'$ be the map sending the ith circle y_i of Y to $x_i(x_i')^{-1}$. (Recall that x_i and x_i' represent the ith circles of X and X', respectively.) Since W has the homotopy type of

$$(M \vee M') \cup \left(\begin{array}{c} m \text{ 2-disks attached along the image of } y_i \text{ under} \\ Y \to X \vee X' \xrightarrow[\mu \vee \mu']{} M \vee M' \end{array} \right)$$

we can see that W_k has the homotopy type of

$$(M \vee M')_k \cup \left(\begin{array}{c} \text{2-disks attached along the image of lifts of } y_i \text{ under} \\ Y_k \to (X \vee X')_k \to (M \vee M')_k \end{array} \right)$$

where $\{(X \vee X')_k\}$ and $\{Y_k\}$ are the pullback p-towers of $X \vee X'$ and Y determined by $\{Z_k\}$, respectively. (In fact, Y_k is a disjoint union of copies of Y since $Y \to Z$ factors through W and $\pi_1(Y) \to \pi_1(W)$ is trivial.) From this we obtain a Mayer-Vietoris exact sequence

$$H_2((M \vee M')_k; \mathbb{Z}_{(p)}) \longrightarrow H_2(W_k; \mathbb{Z}_{(p)}) \longrightarrow H_1(Y_k; \mathbb{Z}_{(p)}) \longrightarrow H_1((M \vee M')_k; \mathbb{Z}_{(p)}).$$

We will complete the proof by showing that the rightmost map on $H_1(-;\mathbb{Z}_{(p)})$ is injective. For this purpose, we factor $Y \to M \vee M'$ into a composition of 3 maps:

$$Y \xrightarrow{\iota} Y \vee Y' \xrightarrow{\alpha} X \vee X' \xrightarrow{\mu \vee \mu'} M \vee M'$$

where $Y' = Y$, the first map ι is an obvious inclusion into the first factor, and the second map α is defined by $y_i \mapsto x_i(x_i')^{-1}$ and $y_i' \mapsto x_i'$ (here y_i' represents the ith circle of Y'). We investigate the induced maps on pullback p-towers $\{Y_k\}$, $\{(Y \vee Y')_k\}$, $\{(X \vee X')_k\}$, and $\{(M \vee M')_k\}$ determined by $\{Z_k\}$:

(1) ι induces an injection

$$H_1(Y_k;\mathbb{Z}_{(p)}) \longrightarrow H_1((Y \vee Y')_k;\mathbb{Z}_{(p)})$$

since $Y_k \to (Y \vee Y')_k$ is an embedding between 1-complexes, being the lift of an embedding ι.

(2) α induces an isomorphism

$$H_1((Y \vee Y')_k;\mathbb{Z}_{(p)}) \longrightarrow H_1((X \vee X')_k;\mathbb{Z}_{(p)}).$$

For, α induces an isomorphism on $\widehat{\pi_1(-)}$ by Lemma 2.6 (1), since α is an H_1-isomorphism and $H_2(X \vee X';\mathbb{Z}_{(p)}) = 0$. By Lemma 2.8 stated below, it follows that

$$(Y \vee Y')_k \longrightarrow (X \vee X')_k$$

induces an isomorphism on $\widehat{\pi_1(-)}$. From this the claim follows, by Lemma 2.6 (0).

(3) $\mu \vee \mu'$ induces an isomorphism

$$H_1((X \vee X')_k;\mathbb{Z}_{(p)}) \longrightarrow H_1((M \vee M')_k;\mathbb{Z}_{(p)}).$$

For, for the CW-complex Z in the proof of the previous lemma, we have a commutative diagram

$$
\begin{array}{ccc}
X \vee X' & \xrightarrow{\mu \vee \mu'} & M \vee M' \\
& \searrow \qquad \swarrow & \\
& Z \vee Z &
\end{array}
$$

with maps into $Z \vee Z$ 2-connected. Applying Lemma 2.6 (1), it follows that $\mu \vee \mu'$ induces an isomorphism on $\widehat{\pi_1(-)}$. Now the claim is shown by Lemma 2.8 and Lemma 2.6 (0) as we did above.

Combining (1), (2), and (3), it follows that

$$H_1(Y_k;\mathbb{Z}_{(p)}) \longrightarrow H_1((M \vee M')_k;\mathbb{Z}_{(p)})$$

is injective. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 2.8. *Suppose A and B are CW-complexes with finite 2-skeletons and $A \to B$ is a map inducing an isomorphism on $\widehat{\pi_1(-)}$. Suppose $\pi_1(B) \to \Gamma$ is a map into an abelian p-group Γ, and A_Γ and B_Γ are covers of A and B, respectively, induced by $\pi_1(A) \to \pi_1(B) \to \Gamma$. Then the lift $A_\Gamma \to B_\Gamma$ induces an isomorphism on $\widehat{\pi_1(-)}$. Consequently, for any p-tower $\{B_k\}$ of B and the pullback p-tower $\{A_k\}$ of A, $A_k \to B_k$ induces an isomorphism on $\widehat{\pi_1(-)}$.*

Proof. By Lemma 2.6 (2), there is a commutative diagram

$$
\begin{array}{ccc}
A & \longrightarrow & B \\
& \searrow \quad \swarrow & \\
& Z &
\end{array}
$$

where Z has finite 2-skeleton and vertical maps are 2-connected. Since $H_1(B; \mathbb{Z}_{p^r}) \cong H_1(Z; \mathbb{Z}_{p^r})$ for all r, the given $\pi_1(B) \to \Gamma$ factors through $\pi_1(Z)$. So, taking Γ-covers A_Γ, B_Γ, and Z_Γ of A, B, and Z, we obtain a commutative diagram

Since Γ is a p-group and the maps into Z in the previous diagram are 2-connected, the maps into Z_Γ in this diagram are 2-connected too, by Levine's result [16, Proof of Proposition 3.2]. (Refer to [2, Lemma 3.2 and 3.3], [6, Corollary 4.13] for statements that apply to our case directly. See also [6, Corollary 4.13].) By Lemma 2.6 (1), it follows that $\widehat{\pi_1(A_\Gamma)} \cong \widehat{\pi_1(Z_\Gamma)} \cong \widehat{\pi_1(B_\Gamma)}$. $\qquad\square$

Now we are ready to prove Theorem 2.4.

Proof of Theorem 2.4. Suppose two $\widehat{F}$-string links β, β' and a p-structure $\mathcal{T} = (\{X_k\}, \theta)$ for X are given. Recall our notation: $(\{M_k\}, \phi)$, $(\{M'_k\}, \phi')$, $(\{N_k\}, \psi)$ are the p-structures of surgery manifolds of β, β', $\beta \cdot \beta'$ which are determined by the given p-structure $\mathcal{T}$, respectively.

We need to show that $\lambda(M_n, \phi) + \lambda(M'_n, \phi') = \lambda(N_n, \psi)$. By Lemma 2.5, there is a bordism W_n endowed with a character $\pi_1(W_n) \to \mathbb{Z}_d$ between $(\mathcal{M}_n, \phi) \cup (M'_n, \phi')$ and (N_n, ψ). By the definition of $\lambda(-, -)$ (see Definition 2.2 in [2]), we have

$$\lambda(M_n, \phi) + \lambda(M'_n, \phi') - \lambda(N_n, \psi) = [\lambda_{\mathbb{Q}(\zeta_d)}(W_n)] - [\lambda_{\mathbb{Q}}(W_n)]$$

where $[\lambda_{\mathbb{Q}(\zeta_d)}(W_n)]$ is the Witt class of (the nonsingular part of) the $\mathbb{Q}(\zeta_d)$-valued intersection form on $H_2(W_n; \mathbb{Q}(\zeta_d))$, and $[\lambda_{\mathbb{Q}}(W_n)]$ is the Witt class of (the nonsingular part of) the ordinary intersection form on $H_2(W_n; \mathbb{Q})$.

By Lemma 2.7,

$$H_2((M \cup M)_n; \mathbb{Z}_{(p)}) \longrightarrow H_2(W_n; \mathbb{Z}_{(p)})$$

is surjective, and so is for $\mathbb{Q}$-coefficients. It follows that $[\lambda_{\mathbb{Q}}(W_n)] = 0$.

Let $(M \cup M')_{n+1}$ be the $\mathbb{Z}_d$-cover of $(M \cup M')_n$ determined by ϕ and ϕ', and denote the $\mathbb{Z}_d$-cover of W_n by W_{n+1} similarly. Note that $H_2(W_n; \mathbb{Q}[\mathbb{Z}_d]) = H_2(W_{n+1}; \mathbb{Q})$ and similarly for $M \cup M'$. So, applying Lemma 2.7 for $k = n + 1$,

$$H_2((M \cup M)_n; \mathbb{Q}[\mathbb{Z}_d]) \longrightarrow H_2(W_n; \mathbb{Q}[\mathbb{Z}_d])$$

is surjective. Since $\mathbb{Q}(\zeta_d)$ is $\mathbb{Q}[\mathbb{Z}_d]$-flat,

$$H_*(-; \mathbb{Q}(\zeta_d)) = H_*(-; \mathbb{Q}[\mathbb{Z}_d]) \otimes_{\mathbb{Q}[\mathbb{Z}_d]} \mathbb{Q}(\zeta_d)$$

and thus the surjectivity on H_2 holds for $\mathbb{Q}(\zeta_d)$-coefficients as well as $\mathbb{Q}[\mathbb{Z}_d]$. It follows that $[\lambda_{\mathbb{Q}(\zeta_d)}(W_n)] = 0$. $\qquad\square$

3. Obstructions to being $(n.5)$-solvable

In this section we prove that the Hirzebruch type invariants of $\widehat{F}$-links give obstructions to being $(n.5)$-solvable, sharpening the results on $(n + 1)$-solvability in [2]. Our result is best described in terms of the following p-analogue of the integral (or rational) (h)-solvability. Denote the nth lower central subgroup of a group G by $G^{(n)}$, i.e., $G^{(0)} = G$ and $G^{(n+1)} = [G^{(n)}, G^{(n)}]$.

Definition 3.1. Suppose M is a closed 3-manifold. A 4-manifold W bounded by M is called a $\mathbb{Z}_{(p)}$-*coefficient* (n)-*solution of* M if the following holds:

    (1) W is a $\mathbb{Z}_{(p)}$-coefficient H_1-bordism, i.e., $H_1(M; \mathbb{Z}_{(p)}) \to H_1(W; \mathbb{Z}_{(p)})$ is an isomorphism.

(2) There exist $u_1, \ldots, u_r, v_1, \ldots, v_r \in H_2(W; \mathbb{Z}_{(p)}[\pi/\pi^{(n)}])$, where $\pi = \pi_1(W)$ and $r = \frac{1}{2}\beta_2(W)$, such that the $\mathbb{Z}_{(p)}[\pi/\pi^{(n)}]$-valued intersection form $\lambda_W^{(n)}$ on $H_2(W; \mathbb{Z}_{(p)}[\pi/\pi^{(n)}])$ satisfies $\lambda_W^{(n)}(u_i, u_j) = 0$ and $\lambda_W^{(n)}(u_i, v_j) = \delta_{ij}$.

If, in addition, the following holds, then W is called a $\mathbb{Z}_{(p)}$-*coefficient* $(n.5)$-*solution* of M:

(3) There exist $\tilde{u}_1, \ldots, \tilde{u}_r \in H_2(W; \mathbb{Z}_{(p)}[\pi/\pi^{(n+1)}])$ such that $\lambda_W^{(n+1)}(\tilde{u}_i, \tilde{u}_j) = 0$ and u_i is the image of $\tilde{u}_i$.

If there is a $\mathbb{Z}_{(p)}$-coefficient (h)-solution of M ($h \in \frac{1}{2}\mathbb{Z}_{\geq 0}$), then M is said to be $\mathbb{Z}_{(p)}$-*coefficient* (h)-*solvable*. A (string) link is called $\mathbb{Z}_{(p)}$-*coefficient* (h)-*solvable* if the surgery manifold (of its closure) is $\mathbb{Z}_{(p)}$-coefficient (h)-solvable.

Obviously an integral (h)-solution defined in [10] is a $\mathbb{Z}_{(p)}$-coefficient (h)-solution. In this section, as an abuse of terminology, an (h)-solution always designates a $\mathbb{Z}_{(p)}$-coefficient (h)-solution, and similarly for an (h)-solvable 3-manifold.

For a given $\phi \colon \pi_1(M) \to \mathbb{Z}_{p^a}$, denote $\Gamma_j = \mathbb{Z}_{p^j}$ ($j = 0, 1, \ldots, a$) and let M_{Γ_j} be the cover of M determined by

$$\pi_1(M) \xrightarrow{\phi} \mathbb{Z}_{p^a} \xrightarrow{\text{proj.}} \mathbb{Z}_{p^a}/p^j \mathbb{Z}_{p^a} = \Gamma_j.$$

Theorem 3.2. *Suppose* $(\{M_i\}, \phi)$ *is a p-structure of height n for M. If M is $(n.5)$-solvable, $H_1(M_i)$ is p-torsion free for all i, and $\beta_1((M_n)_{\Gamma_j}) - 1 = |\Gamma_j|(\beta_1(M_n) - 1)$ for $j = 0, 1, \ldots, a$, then $\lambda(M_n, \phi)$ is well-defined and vanishes.*

Before proving Theorem 3.2, we discuss its consequences for links:

Corollary 3.3. *Suppose* $(\{M_i\}, \phi)$ *is a p-structure of height n for the surgery manifold of an $\widehat{F}$-link L. If L is $(n.5)$-solvable, then $\lambda(M_n, \phi) = 0$.*

Corollary 3.4. *If β is an $(n.5)$-solvable $\widehat{F}$-string link, then for any p-structure $\mathcal{T} = (\{X_i\}, \theta)$ of height n for X, $\lambda_\mathcal{T}(\beta) = 0$.*

Proof of Corollaries 3.3 and 3.4. First note that Corollary 3.4 follows immediately from Corollary 3.3. To prove Corollary 3.3, recall that in the proof of Lemma 2.2 we have shown the following: if $\{M_i\}$ is a p-tower of the surgery manifold M of an $\widehat{F}$-link, then $H_1(M_i)$ is p-torsion free and a meridian map $X = \bigvee^m S^1 \to M$ induces a pullback p-tower $\{X_i\}$ such that $H_1(M_i) \otimes \mathbb{Z}_p \cong H_1(X_i) \otimes \mathbb{Z}_p$. From this it follows that $\beta_1(M_i) = \beta_1(X_i)$. A straightforward Euler characteristic computation (e.g., see [2, Corollary 6.5]) for the cover X_i of X shows that $\beta_1(X_i) - 1 = d(\beta_1(X) - 1)$ when X_i is a d-fold cover of X. It follows that Theorem 3.2 applies to conclude that $\lambda(M_n, \phi) = 0$. $\square$

Recall that $\widehat{\mathcal{C}}^{\mathrm{SL}}$ is the subgroup of the classes of $\widehat{F}$-string links in $\mathcal{C}^{\mathrm{SL}}$, $\mathcal{F}_{(h)}^{\mathrm{SL}}$ is the subgroup of the classes of integrally (h)-solvable string links in $\mathcal{C}^{\mathrm{SL}}$ in the sense of [10, 14], and $\widehat{\mathcal{F}}_{(n)}^{\mathrm{SL}} = \widehat{\mathcal{C}}^{\mathrm{SL}} \cap \mathcal{F}_{(n)}^{\mathrm{SL}}$. Since an integral (h)-solution is a $\mathbb{Z}_{(p)}$-coefficient (h)-solution, as an immediate consequence of Corollary 3.4 and Theorem 2.4 we obtain the first part of Theorem 1.2 in the introduction: $\lambda_\mathcal{T}(-)$ induces a group homomorphism

$$\lambda_\mathcal{T} \colon \widehat{\mathcal{C}}_{(n)}^{\mathrm{SL}} / \widehat{\mathcal{F}}_{(n.5)}^{\mathrm{SL}} \longrightarrow L^0(\mathbb{Q}(\zeta_d))$$

whenever $\mathcal{T}$ is of height n.

The remaining part of this section is devoted to the proof of Theorem 3.2. The proof of Theorem 3.2 consists of two steps; first we investigate the solvability of an abelian p-cover of an (h)-solvable 3-manifold, in order to reduce the general case to the special case of $n = 0$, and then we complete the proof by showing Theorem 3.2 for the case of $n = 0$. In the following two subsections we deal with each step.

3.1. **Solvability of abelian p-covers.** The first step of the proof of Theorem 3.2, namely Theorem 3.5 below, is also interesting on its own. For a space X endowed with a homomorphism $\pi_1(X) \to \Gamma$, we denote the Γ-cover of X by X_Γ.

Theorem 3.5 (Covering Solution Theorem). *Suppose W is an (h)-solution for M with $h \geq 1$, $\phi \colon \pi_1(M) \to \Gamma$ is a homomorphism onto an abelian p-group Γ, and both $H_1(M)$ and $H_1(M_\Gamma)$ are p-torsion free. Then ϕ extends to $\pi_1(W)$, and W_Γ is an $(h-1)$-solution of M_Γ.*

In the proof of Theorem 3.5, we use the following notations and observations. For a fixed subring R of $\mathbb{Q}$ and a 4-manifold W endowed with $\pi_1(W) \to G$, we denote the RG-valued intersection form on $H_2(W; RG)$ by λ_W^G. Here RG is regarded as a ring with involution $\bar{g} = g^{-1}$ as usual. We adopt the convention that $H_*(W; RG)$ is a right RG-module and λ_W^G is defined by

$$\lambda_W^G(x, y) = \sum_{g \in G} (x \cdot yg) g$$

where $\cdot$ designates the ordinary intersection number. Then λ_W^G satisfies $\lambda_W^G(xg, y) = \lambda_W^G(x, y)g$, $\lambda_W^G(x, yg) = \bar{g}\lambda_W^G(x, y)$, and $\lambda_W^G(y, x) = \overline{\lambda_W^G(x, y)}$.

Suppose $G \to \Gamma$ is a surjective group homomorphism with kernel H. Then the Γ-cover W_Γ is defined, and since $\pi_1(W) \to G$ induces $\pi_1(W_\Gamma) \to H$, the H-cover $(W_\Gamma)_H$ of W_Γ is also defined. In fact, $W_G = (W_\Gamma)_H$. It follows that $H_*(W; RG) \cong H_*(W_\Gamma; RH)$. Choose pre-images $g_i \in G$ $(i = 1, \ldots, |\Gamma|)$ of each element of Γ, i.e., the g_i are coset representatives of $H \subset G$. Note that $\lambda_{W_\Gamma}^H$ is the RH-valued intersection form of W_Γ. Regarding RH as a subring of RG, we have the following identity, which can be verified easily by using the defining formula of λ_W^G and $\lambda_{W_\Gamma}^H$.

Lemma 3.6. $\lambda_W^G(x, y) = \sum_i \lambda_{W_\Gamma}^H(x, yg_i) g_i \in RG$

The following observation is also necessary to prove Theorem 3.5.

Lemma 3.7. *Suppose R is a subring of $\mathbb{Q}$. If W is a 4-manifold with boundary M and $H_1(M; R) \to H_1(W; R)$ is surjective, then $H_2(W; R)$ is a free R-module.*

Proof. We have

$$H_2(W; R) = H^2(W, M; R) = \mathrm{Hom}(H_2(W, M), R) \oplus \mathrm{Ext}(H_1(W, M), R).$$

Since R is torsion free, $\mathrm{Hom}(H_2(W, M), R)$ is isomorphic to a free R-module of rank $\beta_2(W)$. By the surjectivity assumption, $0 = H_1(W, M; R) = H_1(W, M) \otimes R$. From this it follows that the free part of $H_1(W, M)$ is trivial. Also, if $H_1(W, M)$ has a $\mathbb{Z}_r$-summand, then $1/r \in R$. Since $\mathrm{Ext}(\mathbb{Z}_r, R) = R/rR$, it follows that $\mathrm{Ext}(H_1(W, M), R) = 0$. $\square$

Proof of Theorem 3.5. Recall our assumptions: W is an (h)-solution of M, $h > 1$, M_Γ is a Γ-cover determined by $\pi_1(M) \to \Gamma$, and both $H_1(M)$ and $H_1(M_\Gamma)$ are p-torsion free.

Since $H_1(M; \mathbb{Z}_{(p)}) \to H_1(W; \mathbb{Z}_{(p)})$ is an isomorphism, M and W have the same $H_1(-)/(\text{torsion coprime to } p)$. Since Γ is an abelian p-group, $\pi_1(M) \to \Gamma$ factors through $H_1(M)/(\text{torsion coprime to } p)$, and thus it extends to $\pi_1(W)$. Therefore, there is defined the Γ-cover W_Γ of W with boundary M_Γ.

Also, $H_1(W, M; \mathbb{Z}_{(p)}) = 0$. It follows that $H_1(W_\Gamma, M_\Gamma; \mathbb{Z}_{(p)}) = 0$ by Levine's argument [16, Proof of Proposition 3.2] (see [2, Lemma 3.2, 3.3], [6, Corollary 4.13] for statements that apply to our case directly), and so $H_1(M_\Gamma; \mathbb{Z}_{(p)}) \to H_1(W_\Gamma; \mathbb{Z}_{(p)})$ is surjective. To show that $H_1(M_\Gamma; \mathbb{Z}_{(p)}) \to H_1(W_\Gamma; \mathbb{Z}_{(p)})$ is an isomorphism, we will prove that in the exact sequence

$$H_2(W_\Gamma; \mathbb{Z}_{(p)}) \longrightarrow H_2(W_\Gamma, M_\Gamma; \mathbb{Z}_{(p)}) \longrightarrow H_1(M_\Gamma; \mathbb{Z}_{(p)}) \longrightarrow H_1(W_\Gamma; \mathbb{Z}_{(p)}) \longrightarrow 0$$

the leftmost map is surjective. For this purpose, we need the following two claims:

Claim 1. $\beta_2(W_\Gamma) \leq |\Gamma|\beta_2(W)$.

To prove this we appeal to the following fact proved in [2]: for any p-group Γ,

$$\beta_2(W_\Gamma; \mathbb{Z}_p) \leq |\Gamma|\beta_2(W; \mathbb{Z}_p)$$

where $\beta_i(-; \mathbb{Z}_p)$ denotes the $\mathbb{Z}_p$-Betti number. In our case, since $H_2(W)$ is p-torsion free by Lemma 3.7, we have

$$\beta_2(W_\Gamma) \leq \beta_2(W_\Gamma; \mathbb{Z}_p) \leq |\Gamma|\beta_2(W; \mathbb{Z}_p) = |\Gamma|\beta_2(W).$$

For the second claim, we write $h = n$ or $n.5$ for some integer n, and let $G = \pi_1(W)/\pi_1(W)^{(n)}$. Note that there is $G \to \Gamma$ since $\pi_1(W)^{(n)} \subset \pi_1(W)^{(1)} \subset \pi_1(W_\Gamma) \subset \pi_1(W)$. As in Lemma 3.6, let $H = \pi_1(W_\Gamma)/\pi_1(W)^{(n)}$ be the kernel of $G \to \Gamma$.

Claim 2. *There are elements* $x_1, \ldots, x_m, y_1, \ldots, y_m$ *in* $H_2(W_\Gamma; \mathbb{Z}_{(p)}H)$ *such that* $\lambda^H_{W_\Gamma}(x_i, x_j) = 0$ *and* $\lambda^H_{W_\Gamma}(x_i, y_j) = \delta_{ij}$, *where* $m = \frac{1}{2}|\Gamma|\beta_2(W)$.

To construct the x_i and y_i, we start with $u_1, \ldots, u_r, v_1, \ldots, v_r \in H_2(W; \mathbb{Z}_{(p)}G)$ such that $\lambda^G_W(u_i, u_j) = 0$ and $\lambda^G_W(u_i, v_j) = \delta_{ij}$, which are given by the definition of the (h)-solvability, where $r = \frac{1}{2}\beta_2(W)$. Choose a pre-image $g_i \in G$ ($i = 1, \ldots, |\Gamma|$) of each element in Γ as in Lemma 3.6. Then the elements $u_i g_k, v_i g_k$ in $H_2(W; \mathbb{Z}_{(p)}G) = H_2(W_\Gamma; \mathbb{Z}_{(p)}H)$ satisfy $\lambda^H_{W_\Gamma}(u_i g_k, u_j g_\ell) = 0$ and $\lambda^H_{W_\Gamma}(u_i g_k, v_j g_\ell) = \delta_{ij}\delta_{k\ell}$ by Lemma 3.6. This proves Claim 2.

Returning to the proof of the surjectivity of $H_2(W_\Gamma; \mathbb{Z}_{(p)}) \to H_2(W_\Gamma, M_\Gamma; \mathbb{Z}_{(p)})$, consider the images $\bar{x}_i, \bar{y}_i \in H_2(W_\Gamma; \mathbb{Z}_{(p)})$ of the x_i and y_i. By Claim 2 and by the naturality of the intersection form, it follows that the $\mathbb{Z}_{(p)}$-valued ordinary intersection numbers of the x_i and y_i in W_Γ are given by $\bar{x}_i \cdot \bar{x}_j = 0$ and $\bar{x}_i \cdot \bar{y}_j = \delta_{ij}$. Therefore $\beta_2(W_\Gamma) \geq \frac{1}{2}|\Gamma|\beta_2(W)$. Combining this with Claim 1, we have $\beta_2(W_\Gamma) = \frac{1}{2}|\Gamma|\beta_2(W)$, and $H_2(W_\Gamma; \mathbb{Q}) \to H_2(W_\Gamma, M_\Gamma; \mathbb{Q})$ is an isomorphism.

Consider the below commutative diagram:

$$
\begin{array}{ccc}
H_1(M_\Gamma; \mathbb{Z}_{(p)}) & \longrightarrow & H_1(W_\Gamma; \mathbb{Z}_{(p)}) \\
\downarrow & & \downarrow \\
H_1(M_\Gamma; \mathbb{Q}) & \longrightarrow & H_1(W_\Gamma; \mathbb{Q})
\end{array}
$$

The bottom horizontal arrow is injective by the last paragraph, and so is the left vertical arrow, since $H_1(M_\Gamma)$ is p-torsion free. It follows that the top horizontal arrow is also injective, and therefore is an isomorphism.

Now, observe that $\pi_1(W)^{(n)} \subset \pi_1(W_\Gamma)^{(n-1)}$ since Γ is abelian. So there is an induced map $H \to N = \pi_1(W_\Gamma)/\pi_1(W_\Gamma)^{(n-1)}$. Let $x'_i, y'_i \in H_2(W_\Gamma; \mathbb{Z}_{(p)}N)$ be the image of $x_i, y_i \in H_2(W_\Gamma; \mathbb{Z}_{(p)}H)$ given in Claim 2 ($i = 1, \ldots, m$). Then by the naturality of the intersection form, $\lambda^N_{W_\Gamma}(x'_i, x'_i) = 0$ and $\lambda^N_{W_\Gamma}(x'_i, y'_i) = \delta_{ij}$. Since $m = \frac{1}{2}|\Gamma|\beta_2(W) = \frac{1}{2}\beta_2(W_\Gamma)$, it follows that W_Γ is an $(h-1)$-solution of M_Γ when $h = n$.

For the case of $h = n.5$, there are $\tilde{u}_1, \ldots, \tilde{u}_r \in H_2(W; \mathbb{Z}_{(p)}[\pi_1(W)/\pi_1(W)^{(n+1)}])$ such that $\lambda^{(n+1)}_W(\tilde{u}_i, \tilde{u}_j) = 0$ and u_i is the image of $\tilde{u}_i$. Similarly to the construction of the above x_i, we can produce m elements $\tilde{x}_i \in H_2(W_\Gamma; \mathbb{Z}_{(p)}[\pi_1(W_\Gamma)/\pi_1(W_\Gamma)^{(n)}])$ from the $\tilde{u}_i$, in such a way that the $\tilde{x}_i$ together with the x_i, y_i satisfy the intersection form condition required in the definition of $(n.5)$-solvability. This completes the proof for $h = n.5$. □

3.2. Vanishing of $\lambda(M, \phi)$ for (0.5)-solvable M.

Theorem 3.8. *Suppose M is (0.5)-solvable 3-manifold with p-torsion free $H_1(M)$ and $\phi\colon \pi_1(M) \to \Gamma = \mathbb{Z}_{p^a}$ is a character such that $\beta_1(M_{\Gamma_i}) - 1 = p^i(\beta_1(M) - 1)$ for $i = 0, 1, \ldots, a$. Then $\lambda(M, \phi) = 0$ in $L^0(\mathbb{Q}(\zeta_{p^a}))$.*

Here M_{Γ_i} is the $\mathbb{Z}_{p^i}$-cover of M defined in the paragraph before Theorem 3.2.

By applying Covering Solution Theorem 3.5 inductively n times and applying Theorem 3.8 to the nth iterated cover, the main theorem of this section (Theorem 3.2) follows immediately.

Proof. Suppose W is a ($\mathbb{Z}_{(p)}$-coefficient) (0.5)-solution of M. Since $H_1(M; \mathbb{Z}_{(p)}) \cong H_1(W; \mathbb{Z}_{(p)})$, ϕ extends to $\pi_1(W)$ as in the proof of Theorem 3.5. Therefore $\lambda(M, \phi)$ is defined and can be computed from the intersection forms of W.

Since W is a (0.5)-solution, there are $\tilde{u}_1, \ldots, \tilde{u}_r \in H_2(W; \mathbb{Z}_{(p)}[\pi/\pi^{(1)}])$ and $v_1, \ldots, u_r \in H_2(W; \mathbb{Z}_{(p)})$ such that $\lambda_W^{(1)}(\tilde{u}_i, \tilde{u}_j) = 0$ and $\lambda_W(u_i, v_j) = \delta_{ij}$ where $\pi = \pi_1(W)$, $r = \frac{1}{2}\beta_2(W)$, and u_i is the image of $\tilde{u}_i$. Multiplying the $\tilde{u}_i$ and v_j by some constant c coprime to p, we may assume that the u_i and v_i are in the image of $H_2(W; \mathbb{Z}[\pi/\pi^{(1)}])$ and $H_2(W)$, at the cost of $\lambda_W(u_i, v_j) = c^2 \delta_{ij}$ instead of δ_{ij}. Choose the pre-images $\tilde{x}_i \in H_2(W; \mathbb{Z}_{(p)}[\pi/\pi^{(1)}])$ of $\tilde{u}_i$ and $y_j \in H_2(W; \mathbb{Z}_{(p)})$ of v_j, respectively.

Denote $\omega_i = \zeta_{p^i}$. Since Γ_i is abelian, there is a canonically induced map $\mathbb{Z}[\pi/\pi^{(1)}] \to \mathbb{Z}\Gamma_i$, and $\mathbb{Z}\Gamma_i$, $\mathbb{Z}_p\Gamma_i$, $\mathbb{Q}\Gamma_i$, and $\mathbb{Q}(\omega_i)$ can be used as homology coefficients of W (and M, (W, M) as well). For $\mathcal{R} = \mathbb{Z}\Gamma_i$, $\mathbb{Z}_p\Gamma_i$, $\mathbb{Q}\Gamma_i$, and $\mathbb{Q}(\omega_i)$, let

$$A(\mathcal{R}) = \mathrm{Im}\{H_2(W; \mathcal{R}) \longrightarrow H_2(W, M; \mathcal{R})\}, \text{ and}$$
$$L(\mathcal{R}) = \mathcal{R}\text{-submodule of } A(\mathcal{R}) \text{ generated by the image of } \tilde{x}_i.$$

Note that Γ_0 is the trivial group so that $\mathbb{Z}\Gamma_0 = \mathbb{Z}$ and similarly for $\mathbb{Z}_p$ and $\mathbb{Q}$ coefficients.

Claim 1. $\dim_{\mathbb{Q}} L(\mathbb{Q}\Gamma_i) \geq \frac{1}{2}\dim_{\mathbb{Q}} A(\mathbb{Q}\Gamma_i)$ *for $i = 0, 1, \ldots, a$.*

To prove Claim 1, we need the following fact, which is a p-analogue of a result due to Cochran and Harvey [7, Proposition 2.7]:

$$(*) \qquad \dim_{\mathbb{Z}_p} \frac{H_2(W, M; \mathbb{Z}_p\Gamma_i)}{L(\mathbb{Z}_p\Gamma_i)} \leq |\Gamma_i| \dim_{\mathbb{Z}_p} \frac{H_2(W, M; \mathbb{Z}_p)}{L(\mathbb{Z}_p)}$$

Since the proof of [7, Proposition 2.7] carries over to our p-group case (using p-analogous statements when necessary; see also [6]), we omit details of the proof of $(*)$.

Since $H_1(M; \mathbb{Z}_{(p)}) \cong H_1(W; \mathbb{Z}_{(p)})$, $H_1(W, M; \mathbb{Z}_{(p)}) = 0$. Consequently $H_1(W, M)$ is p-torsion free. Also, observe that $H_1(W)$ is p-torsion free since $H_1(W; \mathbb{Z}_{(p)}) \cong H_1(M; \mathbb{Z}_{(p)})$ has no torsion by the hypothesis. So,

$$H_2(W, M; \mathbb{Z}_{(p)}) = H^2(W; \mathbb{Z}_{(p)}) = \mathrm{Hom}(H_2(W), \mathbb{Z}_{(p)}) \oplus \mathrm{Ext}(H_1(W), \mathbb{Z}_{(p)})$$
$$= \mathrm{Hom}(H_2(W), \mathbb{Z}_{(p)})$$

has no torsion, and therefore $H_2(W, M)$ is p-torsion free. From these observations, it follows that

$$\dim_{\mathbb{Z}_p} H_2(W, M; \mathbb{Z}_p) = \mathrm{rank}_{\mathbb{Z}} H_2(W, M)/\mathrm{torsion} = \beta_2(W).$$

Recall that $L(\mathbb{Z}_p)$ is generated by the images of the $\frac{1}{2}\beta_2(W)$ elements $\tilde{x}_i$. Since there are the dual elements, namely the images of the y_j whose intersection with the image $\tilde{x}_i$ is $c^2 \delta_{ij}$, and since our constant c is coprime to p, it follows that the

images of the $\tilde{x}_i$ are $\mathbb{Z}_p$-linearly independent in $H_2(W, M; \mathbb{Z}_p)$ so that $\dim_{\mathbb{Z}_p} L(\mathbb{Z}_p) = \frac{1}{2}\beta_2(W)$. Therefore, by $(*)$, we have

$$(**) \qquad\qquad \dim_{\mathbb{Z}_p} \frac{H_2(W, M; \mathbb{Z}_p \Gamma_i)}{L(\mathbb{Z}_p \Gamma_i)} \le \frac{|\Gamma_i|}{2}\beta_2(W).$$

By Levine's result [16, Proof of Proposition 3.2] (refer to [2, Lemma 3.2 and 3.3], [6, Corollary 4.13] for statements that apply to our case directly), $H_1(W, M; \mathbb{Z}_{(p)}) = 0$ implies that $H_1(W, M; \mathbb{Z}_{(p)}\Gamma_i) = 0$. So $H_1(W, M; \mathbb{Z}\Gamma_i)$ is p-torsion free, and

$$H_2(W, M; \mathbb{Z}_p\Gamma) = H_2(W, M; \mathbb{Z}\Gamma) \otimes_{\mathbb{Z}} \mathbb{Z}_p.$$

Since $\otimes_{\mathbb{Z}} \mathbb{Z}_p$ preserves cokernels, we have

$$\begin{aligned}
\frac{H_2(W, M; \mathbb{Z}_p\Gamma_i)}{L(\mathbb{Z}_p\Gamma_i)} &= \operatorname{Coker}\{L(\mathbb{Z}\Gamma_i) \otimes_{\mathbb{Z}} \mathbb{Z}_p \longrightarrow H_2(W, M; \mathbb{Z}\Gamma_i) \otimes_{\mathbb{Z}} \mathbb{Z}_p\} \\
&= \operatorname{Coker}\{L(\mathbb{Z}\Gamma_i) \longrightarrow H_2(W, M; \mathbb{Z}\Gamma_i)\} \otimes_{\mathbb{Z}} \mathbb{Z}_p \\
&= \frac{H_2(W, M; \mathbb{Z}\Gamma_i)}{L(\mathbb{Z}\Gamma_i)} \otimes_{\mathbb{Z}} \mathbb{Z}_p
\end{aligned}$$

and similarly

$$\frac{H_2(W, M; \mathbb{Q}\Gamma_i)}{L(\mathbb{Q}\Gamma_i)} = \frac{H_2(W, M; \mathbb{Z}\Gamma_i)}{L(\mathbb{Z}\Gamma_i)} \otimes_{\mathbb{Z}} \mathbb{Q}.$$

By $(**)$, it follows that

$$(***) \qquad \dim_{\mathbb{Q}} \frac{H_2(W, M; \mathbb{Q}\Gamma_i)}{L(\mathbb{Q}\Gamma_i)} \le \dim_{\mathbb{Z}_p} \frac{H_2(W, M; \mathbb{Z}_p\Gamma_i)}{L(\mathbb{Z}_p\Gamma_i)} \le \frac{|\Gamma_i|}{2}\beta_2(W).$$

Since $H_1(W, M; \mathbb{Z}_{(p)}\Gamma_i) = 0$ as shown above, we have an exact sequence

$$0 \longrightarrow A(\mathbb{Q}\Gamma_i) \longrightarrow H_2(W, M; \mathbb{Q}\Gamma_i) \longrightarrow H_1(M; \mathbb{Q}\Gamma_i) \longrightarrow H_1(W; \mathbb{Q}\Gamma_i) \longrightarrow 0$$

and from this it follows that

$$\dim_{\mathbb{Q}} A(\mathbb{Q}\Gamma_i) = \beta_2(W_{\Gamma_i}) - \beta_1(M_{\Gamma_i}) + \beta_1(W_{\Gamma_i}).$$

Note that $\beta_1(M) = \beta_1(W)$, and by our hypothesis, $\beta_1(M_{\Gamma_i}) = |\Gamma_i|(\beta_1(M) - 1) + 1$. Combining this with $(***)$, we obtain

$$\begin{aligned}
2\dim_{\mathbb{Q}} L(\mathbb{Q}\Gamma_i) &- \dim_{\mathbb{Q}} A(\mathbb{Q}\Gamma_i) \\
&\ge (\beta_2(W_{\Gamma_i}) - \beta_1(W_{\Gamma_i}) + 1) - |\Gamma_i|(\beta_2(W) - \beta_1(W) + 1) \\
&= \chi(W_{\Gamma_i}) - |\Gamma_i|\chi(W) = 0
\end{aligned}$$

since $\beta_3(W) = \beta_1(W, M) = 0$ and $\beta_3(W_{\Gamma_i}) = 0$ similarly. This finishes the proof of Claim 1.

Claim 2. $\dim_{\mathbb{Q}(\omega_i)} L(\mathbb{Q}(\omega_i)) = \frac{1}{2}\dim_{\mathbb{Q}(\omega_i)} A(\mathbb{Q}(\omega_i))$ *for* $i = 0, 1, \ldots, a$.

To prove this, we use an induction on i. First observe that $\le$ always holds since the $\mathbb{Q}(\omega_i)$-valued intersection form on $A(\mathbb{Q}(\omega_i))$ is nonsingular but vanishes on $L(\mathbb{Q}(\omega_i))$. So it suffices to show $\ge$. For $i = 0$, this is none more than Claim 1 for $i = 0$.

Suppose $k > 0$ and that Claim 2 holds for all $i < k$. Recall that the regular representation $\mathbb{Q}[\mathbb{Z}_d]$ is isomorphic to the orthogonal sum $\bigoplus_{r|d} \mathbb{Q}(\zeta_r)$. It follows that for any $j < i$, $\mathbb{Q}(\omega_j)$ are $(\mathbb{Q}\Gamma_i)$-flat. So, we have

$$\begin{aligned}
H_2(W; \mathbb{Q}(\omega_j)) &= H_2(W; \mathbb{Q}\Gamma_i) \otimes_{\mathbb{Q}\Gamma_i} \mathbb{Q}(\omega_j), \\
H_2(W, M; \mathbb{Q}(\omega_j)) &= H_2(W, M; \mathbb{Q}\Gamma_i) \otimes_{\mathbb{Q}\Gamma_i} \mathbb{Q}(\omega_j),
\end{aligned}$$

and therefore

$$A(\mathbb{Q}(\omega_j)) = A(\mathbb{Q}\Gamma_i) \otimes_{\mathbb{Q}\Gamma_i} \mathbb{Q}(\omega_j) \quad \text{and} \quad L(\mathbb{Q}(\omega_j)) = L(\mathbb{Q}\Gamma_i) \otimes_{\mathbb{Q}\Gamma_i} \mathbb{Q}(\omega_j).$$

Using this, we have

$$L(\mathbb{Q}\Gamma_k) = L(\mathbb{Q}\Gamma_k) \underset{\mathbb{Q}\Gamma_k}{\otimes} \mathbb{Q}\Gamma_i = L(\mathbb{Q}\Gamma_k) \underset{\mathbb{Q}\Gamma_k}{\otimes} \left(\bigoplus_{j=0}^{k} \mathbb{Q}(\omega_i) \right)$$

$$= \bigoplus_{j=0}^{k} L(\mathbb{Q}\Gamma_k) \underset{\mathbb{Q}\Gamma_k}{\otimes} \mathbb{Q}(\omega_i) = \bigoplus_{j=0}^{k} L(\mathbb{Q}(\omega_j))$$

and so

$$\dim_{\mathbb{Q}} L(\mathbb{Q}\Gamma_k) = \sum_{j=0}^{k} [\mathbb{Q}(\omega_j) : \mathbb{Q}] \dim_{\mathbb{Q}(\omega_j)} L(\mathbb{Q}(\omega_j)).$$

Similarly, we have an analogous formula for $\dim_{\mathbb{Q}} A(\mathbb{Q}\Gamma_k)$, which is obtained by replacing every occurrence of $L(-)$ with $A(-)$. By Claim 1 for $i = k$ and by the induction hypothesis, we have

$$\dim_{\mathbb{Q}(\omega_k)} L(\mathbb{Q}(\omega_k)) \geq \frac{1}{2} \dim_{\mathbb{Q}(\omega_k)} A(\mathbb{Q}(\omega_k))$$

as desired. This completes the proof of Claim 2.

Now, the nonsingular part of the $\mathbb{Q}(\omega_i)$-valued intersection form $\lambda_{\mathbb{Q}(\omega_i)}(W)$, which is defined on $A(\mathbb{Q}(w_i))$, vanishes on a half-dimensional subspace, namely $L(\mathbb{Q}(w_i))$. It follows that the Witt class $[\lambda_{\mathbb{Q}(\omega_i)}(W)]$ is trivial in $L^0(\mathbb{Q}(\omega_i))$ for $i = 0, 1, \ldots, a$. Denoting the natural map $L^0(\mathbb{Q}) \to L^0(\mathbb{Q}(\zeta_{p^a}))$ by i_*, by definition we have

$$\lambda(M, \phi) = [\lambda_{\mathbb{Q}(\zeta_{p^a})}(W)] - i_*[\lambda_{\mathbb{Q}}(W)] = 0. \qquad \square$$

4. LOCAL KNOTS

In order to investigate the structure of link concordance modulo knot concordance, we consider the quotient of string link concordance group by "local knots"; Figure 2 illustrates the effect of adding a local knot to a string link.
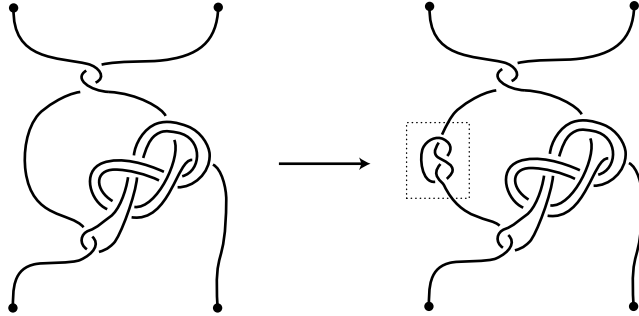

FIGURE 2. Adding a local knot

It can be seen easily that adding a local knot to an m-string link is equivalent to multiplication by a string link β which is the completely split union of one (possibly knotted) arc and $m - 1$ unknotted line segments. Possibly as a slight abuse of terminology, we call β a *local knot*. Note that a local knot β commutes with any string link, so that any subgroup generated by a collection of local knots is normal.

In this section we show that if a p-structure $\mathcal{T}$ is locally trivial, our invariant $\lambda_{\mathcal{T}}$ ignores the effect of adding/removing local knots. Recall from the introduction that $\mathcal{T} = (\{X_i\}, \phi)$ of height n is called *locally trivial* if $\phi(z) = 0$ whenever z is

an element in $\pi_1(X_n)$ which projects to a conjugate of $[x_i]^r$ in $\pi_1(X)$ for some r. (Here x_i denotes the ith circle of X.)

Remark 4.1. We often view a character (into an abelian group) as a homomorphism of H_1, so that $\theta([\gamma])$ is well-defined for the homology class $[\gamma]$ of a loop γ based at any point in X_n. Then it can be seen easily that $\mathcal{T}$ is locally trivial if and only if $\theta([\gamma]) = 0$ whenever γ is a loop in X_n whose projection in X is freely homotopic to x_i^r for some r. Moreover, since a free homotopy in X lifts to X_n and θ is preserved by free homotopy, "freely homotopic to" in the above statement can be omitted.

Theorem 4.2. *If a p-structure $\mathcal{T}$ for X is locally trivial, then $\lambda_{\mathcal{T}}(\beta) = 0$ for any local knot β.*

As an immediate consequence of Theorems 2.4, 3.2, and 4.2, we obtain Addendum to Theorem 1.1 and the second part of Theorem 1.2: If $\mathcal{T}$ is locally trivial, then $\lambda_{\mathcal{T}}$ induces a homomorphism

$$\lambda_{\mathcal{T}} : \frac{\widehat{\mathcal{C}}^{\mathrm{SL}}}{\langle \text{local knots} \rangle} \longrightarrow L^0(\mathbb{Q}(\zeta_d)).$$

If, in addition, $\mathcal{T}$ is of height n, then $\lambda_{\mathcal{T}}$ induces a homomorphism

$$\lambda_{\mathcal{T}} : \frac{\widehat{\mathcal{F}}^{\mathrm{SL}}_{(n)}}{\widehat{\mathcal{F}}^{\mathrm{SL}}_{(n.5)} \cdot \langle \text{local knots} \rangle} \subset \frac{\widehat{\mathcal{C}}^{\mathrm{SL}}}{\widehat{\mathcal{F}}^{\mathrm{SL}}_{(n.5)} \cdot \langle \text{local knots} \rangle} \longrightarrow L^0(\mathbb{Q}(\zeta_d)).$$

To prove Theorem 4.2, we appeal to the following result, which will also be used in later sections as our main computation tool for 3-manifolds and (string) links which are obtained by *infection*. Suppose M is a 3-manifold and α is a simple closed curve in M. Removing an open tubular neighborhood of α from M and then filling it in with the exterior of a knot K in S^3 so that the meridian and preferred longitude of K is identified with a longitude and meridian of α, respectively, we obtain a new 3-manifold N. We say that N is obtained from M by infection along α using K. In [2, Proposition 4.8], it was shown that there is a p-tower map $N \to M$ which extends the identity map between $N - (\text{exterior of } K)$ and $M - (\text{tubular neighborhood of } \alpha)$.

Proposition 4.3 (Corollary 4.7 of [2])**.** *Suppose N is obtained from M by infection along a simple closed curve α using a knot K. Suppose $(\{M_i\}, \phi)$ is a $\mathbb{Z}_d$-valued p-structure of height n and let $(\{N_i\}, \psi)$ be the induced p-structure via pullback along the p-tower map $N \to M$. Let $\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots \subset M_n$ be the components of the pre-image of $\alpha \subset M$, and r_j be the degree of the covering map $\tilde{\alpha}_j \to \alpha$. Then*

$$\lambda(N_n, \psi) = \lambda(M_n, \phi) + \sum_j \left([\lambda_{r_j}(A, \zeta_d^{\phi([\tilde{\alpha}_j])})] - [\lambda_{r_j}(A, 1)] \right)$$

where $[\lambda_r(A, \omega)]$ is the Witt class of (the nonsingular part of) the hermitian form represented by the following $r \times r$ block matrix:

$$\lambda_r(A, \omega) = \begin{bmatrix} A + A^T & -A & & -\omega^{-1}A^T \\ -A^T & A + A^T & \ddots & \\ & \ddots & \ddots & -A \\ -\omega A & & -A^T & A + A^T \end{bmatrix}_{r \times r}$$

For $r = 1$ and 2, $\lambda_r(A, \omega)$ should be understood as

$$\begin{bmatrix} (1 - \omega)A + (1 - \omega^{-1})A^T \end{bmatrix} \quad and \quad \begin{bmatrix} A + A^T & -A - \omega^{-1}A^T \\ -A^T - \omega A & A + A^T \end{bmatrix}.$$

Performing infection on the surgery manifold M_β of a string link β along an unknotted curve α in $D^2 \times [0,1] - \beta$ in such a way that the meridian of K is identified with the preferred longitude of α, the resulting manifold is the surgery manifold M'_β of a new string link β', which is said to be obtained from β by infection. Obviously, the composition of the preferred meridian map $X \to M_{\beta'}$ of β' and the p-tower map $M_{\beta'} \to M_\beta$ is exactly the preferred meridian map of β. So we have a formula relating $\lambda_{\mathcal{T}}(\beta')$ to $\lambda_{\mathcal{T}}(\beta)$ and invariants of K, similarly to Proposition 4.3.

Proof of Theorem 4.2. Suppose $\mathcal{T} = (\{X_i\}, \phi)$ is a $\mathbb{Z}_d$-valued p-structure of height n for X. Observe that a local knot β is obtained from the trivial string link by infection along the ith meridian for some i. Since the trivial link has vanishing $\lambda_{\mathcal{T}}$, by Proposition 4.3 we have

$$\lambda_{\mathcal{T}}(\beta) = \sum_j \left([\lambda_{r_j}(A, \zeta_d^{\phi([\tilde{\alpha}_j])})] - [\lambda_{r_j}(A, 1)] \right)$$

where $\tilde{\alpha}_j$ is a loop in X_n which is a lift of $x_i^{r_j}$ and A is a Seifert matrix of the infection knot. Since $\mathcal{T}$ is locally trivial, $\phi([\tilde{\alpha}_j]) = 0$ for all j. It follows that $\lambda_{\mathcal{T}}(\beta) = 0$. $\square$

For later use, we observe that the defining condition of a locally trivial p-structure is preserved by pullback, as stated below:

Lemma 4.4. *Suppose $f \colon X \to Y$ is a map into a CW-complex Y and $\mathcal{T} = (\{X_k\}, \theta)$ is a p-structure for X induced by a p-structure $(\{Y_k\}, \phi)$ of height n for Y via pullback along f. Then $\mathcal{T}$ is locally trivial if and only if $\phi([\delta]) = 0$ whenever δ is a loop in Y_n whose projection in Y is freely homotopic to $f(x_i^r)$ for some r.*

Proof. Let $f_n \colon X_n \to Y_n$ be the lift of f. Note that a loop δ in Y_n projects to $f(x_i^r)$ if and only if $\delta = f_n(\gamma)$ for some a loop γ in X_n which projects to x_i^r. (For the only if part we need that X_n is the pullback of Y_n.) Also, if a loop δ in Y_n is freely homotopic to $f_n(\gamma)$ for some loop γ in X_n, then $\phi([\delta]) = \theta([\gamma])$ since θ is induced by ϕ. The conclusion follows from this. $\square$

5. Construction of examples

In [2], it was proved that the nth iterated Bing doubles (which has 2^n components) of certain knots are nontrivial in $\mathcal{F}^{\mathrm{L}}_{(n)}/\mathcal{F}^{\mathrm{L}}_{(n+1)}$ but have vanishing Harvey's ρ_n-invariant. In this section, for any m we construct infinitely many m-component string links with similar properties, and furthermore we show that they are linearly independent in the abelianization of $\mathcal{F}^{\mathrm{L}}_{(n)}/\mathcal{F}^{\mathrm{L}}_{(n.5)}$.

Our examples are constructed by infection on a trivial string link. First we show that the class of $\widehat{F}$-(string) links is closed under infection.

Proposition 5.1. *Any (string) link obtained from an $\widehat{F}$-(string) link via infection by a knot is an $\widehat{F}$-(string) link.*

Proof. It suffices to prove the conclusion in the case of a link. Suppose L' is obtained from a link L via by infection along α using a knot. Let E and E' be the exteriors of L and L', respectively. The p-tower map $f \colon E' \to E$ given by [2, Proposition 4.8] is actually an integral homology equivalence preserving the peripheral structure (see the proof of [2, Proposition 4.8]). Choose a meridian map $\mu \colon X = \bigvee^m S^1 \to E$, i.e., μ sends the ith circle of X to a ith meridian of L. We may assume that the image of μ is disjoint from a tubular neighborhood of α so that μ gives rise to a meridian map $\mu' \colon X \to E'$ satisfying $f \circ \mu' = \mu$. Since L is an $\widehat{F}$-link, μ induces an isomorphism on $\widehat{\pi_1(-)}$ [15]. Also, so does f by Lemma 2.6 (1), since f is a homology equivalence. It follows that μ' induces an isomorphism on $\widehat{\pi_1(-)}$. Since

the longitudes of L are killed in $\widehat{\pi_1(E)}$ and f preserves the peripheral structure, the longitudes of L' is killed in $\widehat{\pi_1(E')} \cong \widehat{\pi_1(E)}$. Therefore L' is an $\widehat{F}$-link. $\qquad\square$

We will use infection knots and curves obtained by applying the following two lemmas. In the first lemma, we denote the Levine-Tristram signature of K by $\sigma_K(\omega)$, i.e., for $\omega \in S^1 \subset \mathbb{C}$, $\sigma_K(\omega)$ is defined to be the average of the two one-sided limits of the ω-*signature of K* given by

$$\operatorname{sign}[(1 - \omega)A + (1 - \omega^{-1})A^T]$$

where A is a Seifert matrix of K. We remark two facts: first, one should think of the average in order to obtain a concordance invariant for any $\omega \in S^1$. Second, when ω is a *primitive* p^ath root of unity for some prime power p^a, it is known that there is no nontrivial jump of the ω-signature at ω, so that the average $\sigma_K(\omega)$ is equal to the ω-signature. In this case we do not distinguish $\sigma_K(\omega)$ and the ω-signature.

Lemma 5.2. *There is an infinite sequence $\{K_i\}$ of knots together with a strictly increasing sequence $\{d_i\}$ of powers of p satisfying the following properties:*

(1) $\sigma_{K_i}(\zeta_{d_i}) > 0$, *and if $p = 2$ then $\sigma_{K_i}(\zeta_{d_i}^s) \geq 0$ for any s.*
(2) *if $i < j$, $\sigma_{K_j}(\zeta_{d_i}^s) = 0$ for any s.*
(3) $\int_{S^1} \sigma_{K_i}(\omega) \, d\omega = 0$.
(4) K_i *has vanishing Arf invariant.*

We remark that the conclusion of Lemma 5.2 is slightly stronger than what we actually need in this section; the extra parts will be used in a later section.

Lemma 5.3. *For any $m > 1$ and n, there exist a loop α in $X = \bigvee^m S^1$ and a p-tower $\{X_k\}$ of height n for X satisfying the following:*

(1) $[\alpha] \in \pi_1(X)^{(n)}$ *and every lift $\tilde{\alpha}_j$ of α in X_n is a loop.*
(2) *There is a map $f \colon \pi_1(X_n) \to \mathbb{Z}$ which sends (the class of) each $\tilde{\alpha}_j$ to -1, 0, or 1 and sends at least one $\tilde{\alpha}_j$ to 1. In addition, for any $\mathbb{Z} \to \mathbb{Z}_d$, the composition $\theta \colon \pi_1(X_n) \xrightarrow{f} \mathbb{Z} \to \mathbb{Z}_d$ gives a locally trivial p-structure $(\{X_i\}, \theta)$.*

We postpone the proof of Lemmas 5.2 and 5.3, and proceed to discuss how our examples are constructed.

Suppose $\{K_i\}$ satisfies Lemma 5.2 and α satisfies Lemma 5.3. Let $\mu \colon X \to E_{\beta_0}$ be the preferred meridian map of a trivial string link β_0. Choose any simple closed curve in E_{β_0} which is unknotted in $D^2 \times [0, 1]$ and realizes the homotopy class of $\mu_*([\alpha]) \in \pi_1(E_{\beta_0})$. Let $\beta(K_i)$ be the string link obtained from β_0 via infection using K_i along the chosen simple closed curve. Note that $\beta(K_i)$ is always an $\widehat{F}$-string link by Proposition 5.1.

Theorem 5.4. *The string links $\beta(K_i)$ are (n)-solvable and linearly independent in the abelianization of $\widehat{\mathcal{C}}^{\mathrm{SL}}/(\widehat{\mathcal{F}}^{\mathrm{SL}}_{(n.5)} \cdot \langle local\ knots \rangle)$. Consequently, the abelianization of $\widehat{\mathcal{F}}^{\mathrm{SL}}_{(n)}/(\widehat{\mathcal{F}}^{\mathrm{SL}}_{(n.5)} \cdot \langle local\ knots \rangle)$ is of infinite rank for any n.*

Proof. $\beta(K_i)$ is an $\widehat{F}$-string link by Proposition 5.1. Since $[\alpha] \in \pi_1(X)^{(n)}$ by Lemma 5.3 (1) and K_i has vanishing Arf invariant by Lemma 5.2 (4), $\beta(K_i)$ is (n)-solvable by [10]. So (the concordance class of) $\beta(K_i)$ is in $\widehat{\mathcal{F}}^{\mathrm{SL}}_{(n)}$.

In order to show the independence, we use the invariant $\lambda_{\mathcal{T}}$. Let $\{X_k\}$ be the p-tower in Lemma 5.3, and let $\theta_d \colon \pi_1(X_n) \to \mathbb{Z}_d$ be the composition of the map $f \colon \pi_1(X_n) \to \mathbb{Z}$ in Lemma 5.3 and the projection $\mathbb{Z} \to \mathbb{Z}_d$ sending $1 \in \mathbb{Z}$ to $1 \in \mathbb{Z}_d$. Then, for $\mathcal{T} = (\{X_k\}, \theta_d)$ and a knot K, appealing to Proposition 4.3 we have

$$\lambda_{\mathcal{T}}(\beta(K)) = \sum_j \left([\lambda_1(A, \zeta_d^{\theta_d([\tilde{\alpha}_j])})] - [\lambda_1(A, 1)]\right)$$

where A is a Seifert matrix of K. Observe that $\operatorname{sign}\lambda_1(A,\omega) = \sigma_K(\omega)$, $\sigma_K(1) = 0$, and $\sigma_K(\omega) = \sigma_K(\omega^{-1})$. So by Lemma 5.3 (2),

$$\operatorname{sign}\lambda_{\mathcal{T}}(\beta(K)) = c \cdot \sigma_K(\zeta_d).$$

where c is the number of lifts $\tilde{\alpha}_j$ sent to ± 1 by $f\colon \pi_1(X_n) \to \mathbb{Z}$. Note that $c > 0$ and c is independent of d.

Suppose $\sum_i a_i\beta(K_i) = 0$ in the abelianization of $\widehat{\mathcal{C}}^{\mathrm{SL}}/(\widehat{\mathcal{F}}^{\mathrm{SL}}_{(n.5)} \cdot \langle\text{local knots}\rangle)$ where not all a_i are zero. Choose a minimal i_0 such that $a_{i_0} \neq 0$. Let d_i be as in Lemma 5.2, and let $\mathcal{T} = (\{X_k\}, \phi_{d_{i_0}})$. Note that $\mathcal{T}$ is locally trivial by Lemma 5.3 (2). Then by Theorem 1.2, Lemma 5.2, and by our choice of i_0, we have

$$0 = \operatorname{sign}\lambda_{\mathcal{T}}\left(\sum_i a_i\beta(K_i)\right) = \sum_i a_i \operatorname{sign}\lambda_{\mathcal{T}}(\beta(K_i))$$

$$= \sum_{i \geq i_0} ca_i \cdot \sigma_{K_i}(\zeta_{d_{i_0}}) = ca_{i_0} \cdot \sigma_{K_{i_0}}(\zeta_{d_{i_0}}).$$

Since c and $\sigma_{K_{i_0}}(\zeta_{d_{i_0}})$ is nonzero, a_{i_0} should be zero. From this contradiction, it follows that the $\beta(K_i)$ are linearly independent in the abelianization of $\widehat{\mathcal{C}}^{\mathrm{SL}}/(\widehat{\mathcal{F}}^{\mathrm{SL}}_{(n.5)} \cdot \langle\text{local knots}\rangle)$. $\qquad\square$

Since $\int_{S^1} \sigma_{K_i}(\omega)\,d\omega = 0$ by Lemma 5.2 (3), it follows that each $\beta(K_i)$ has vanishing Harvey's ρ_n-invariant by results in [14]. Also, note that each $\beta(K_i)$ is a boundary link. (In fact it can be seen that any link obtained from a boundary link by infection is again a boundary link.) So, as an immediate consequence of Theorem 5.4, we obtain Theorem 1.3: the abelianization of the kernel of Harvey's homomorphism

$$\rho_n\colon \frac{\mathcal{BF}^{\mathrm{SL}}_{(n)}}{\mathcal{BF}^{\mathrm{SL}}_{(n.5)} \cdot \langle\text{local knots}\rangle} \longrightarrow \mathbb{R}$$

has infinite rank.

5.1. **Construction of infection knots.** In this section we will prove Lemma 5.2. For this purpose, we need the following known facts. The first is a formula for the signature of cable knots.

Lemma 5.5 (Repametrization formula [17, 9, 4, 3]). *For a knot K, let K' be the $(r,1)$-cable of K. Then $\sigma_{K'}(\omega) = \sigma_K(\omega^r)$ for any $\omega \in S^1$.*

The second is a realization result of a "bump" signature function.

Lemma 5.6. *For any $\theta_0 \in (0,\pi)$, there is a knot K and an arbitrarily small neighborhood I of θ_0 contained in $(0,\pi)$ such that $\sigma_K(e^{\theta_0\sqrt{-1}}) \neq 0$ and $\sigma_K(e^{t\sqrt{-1}}) = 0$ for $t \in [0,\pi] - I$.*

Proof. In [5, Proof of Theorem 1], the following statement was shown: for any given $\theta_0 \in (0,\pi)$, there are $\theta \in (0,\theta_0)$ arbitrarily close to θ_0 and a knot K such that $\sigma_K(e^{t\sqrt{-1}}) = 0$ for $0 \leq t < \theta$ and $\sigma_K(e^{t\sqrt{-1}})$ is a nonzero constant for $\theta < t \leq \pi$. Since the Levine-Tristram signature is additive under connected sum, i.e., $\sigma_{K\#J}(\omega) = \sigma_K(\omega) + \sigma_J(\omega)$, our conclusion follows immediate by applying the above statement twice. $\qquad\square$

Proof of Lemma 5.2. We use the following notations: for $\omega \in S^1$, $N_\epsilon(\omega)$ denotes the ϵ-neighborhood of ω in S^1 (with respect to the arc length metric) and $\arg(\omega) \in [0,2\pi)$ denotes the argument of ω.

Suppose θ_0 and θ_1 satisfying $0 < \theta_1 < \theta_0 < \pi/2$ are given. Let ω_0 and ω_1 be points in S^1 such that $\theta_i = \arg(\omega_i)$. We claim that there is a knot K with the following properties:

(1) $\sigma_K(\omega_1) \neq 0$.

(2) $\sigma_K(\omega) = 0$ whenever ω, $-\omega$, $\bar{\omega}$, and $-\bar{\omega}$ are not in I, where

$$I = \left\{ \omega \;\middle|\; \frac{\theta_1}{3} < \arg(\omega) < \theta_0 \right\}.$$

(3) $\int_{S^1} \sigma_K(\omega) \, d\omega = 0$.

To prove the claim, choose $\epsilon > 0$ such that $\epsilon < \min\{\theta_0 - \theta_1, \theta_1/3\}$. By Lemma 5.6, there is a knot J such that $\sigma_J(\omega_1) > 0$ and $\sigma_J(\omega) = 0$ for $\omega \in S^1 - [N_\epsilon(\omega_1) \cup N_\epsilon(\bar{\omega}_1)]$. Let J' be the $(2,1)$-cable of J. Then by Lemma 5.5, $\sigma_{J'}(\omega) = \sigma_J(\omega^2)$. It follows that $\sigma_{J'}(\omega) = 0$ for

$$\omega \in S^1 - \left[N_{\epsilon/2}(\sqrt{\omega_1}) \cup N_{\epsilon/2}(-\sqrt{\omega_1}) \cup N_{\epsilon/2}(\overline{\sqrt{\omega_1}}) \cup N_{\epsilon/2}(-\overline{\sqrt{\omega_1}}) \right]$$

where $\sqrt{\omega_1} = e^{\theta_1 \sqrt{-1}/2}$. Let $K = J \# - J'$. Then, since $\epsilon/2 < \theta_1/2$, $\sigma_{J'}(\omega_1) = 0$ and so $\sigma_K(\omega_1) = \sigma_J(\omega_1) > 0$. From this (1) follows. Since

$$\theta_1/3 < \theta_1/2 - \epsilon/2 < \theta_1 + \epsilon < \theta_0,$$

both $N_\epsilon(\omega_1)$ and $N_{\epsilon/2}(\sqrt{\omega_1})$ are contained in I. From this (2) follows. Since $\int_{S^1} \sigma_{J'}(\omega) \, d\omega = \int_{S^1} \sigma_J(\omega) \, d\omega$, (3) follows.

Now, choose powers $d_1, d_2, \ldots$ of p such that $d_1 \geq 4$ and $d_{i+1} > 3d_i$. For each i, applying the above claim to $\theta_0 = 2\pi/3d_{i-1}$ and $\theta_1 = 2\pi/d_i$, choose inductively a knot K_i which satisfies the above (1), (2), and (3). Replacing K_i by $K_i \# K_i$ if necessary, we may assume that K_i has vanishing Arf invariant. Then it can be checked easily that $\{d_i\}$ and $\{K_i\}$ satisfy Lemma 5.2. $\square$

5.2. Construction of an infection curve and an associated p-structure.

In this subsection we will prove Lemma 5.3. Observe that we may assume that $m = 2$, i.e., $X = S^1 \vee S^1$; once this special case is proved, the general case of $m > 2$ follows easily by attaching $m - 2$ additional circles.

We begin with a description of a loop α in X (which is determined by n). Define loops α_k and β_k in X inductively as follows: let $\alpha_0 = x_0$ and $\beta_0 = x_1$, i.e., the paths representing the two (oriented) 1-cells of X, and

$$\alpha_{k+1} = (\alpha_k, \beta_k) = \alpha_k \beta_k \alpha_k^{-1} \beta_k^{-1}, \quad \beta_{k+1} = \alpha_k (\alpha_k, \beta_k) \alpha_k^{-1}.$$

Note that $\alpha_n, \beta_n \in \pi_1(X)^{(n)}$. Let $\alpha = \alpha_n$.

Next, we construct a p-tower $\{X_k\}$ of height n and investigate the behaviour of the lifts $\tilde{\alpha}_j$ of α in X_n, similarly to [2, Section 7.1]. Fix a power $q > 2$ of p, and let $\Gamma = \mathbb{Z}_q \oplus \mathbb{Z}_q$. Let $X_0 = X$, and let c_0, d_0 be the two (oriented) 1-cells of X_0 corresponding to α_0 and β_0, respectively.

Suppose a cover X_k of X has been defined and two 1-cells c_k and d_k of X_k has been chosen. Define a map $X_k \to K(\Gamma, 1)$ by sending all 0-cells and 1-cells of X_k except c_k, d_k to a basepoint $* \in K(\Gamma, 1)$ and sending c_k, d_k to loops representing $(1,0)$, $(0,1) \in \Gamma$, respectively, and let $\phi_k \colon \pi_1(X_k) \to \Gamma$ be the map induced by this. We define X_{k+1} to be the regular cover of X_k determined by ϕ_k.

X_{k+1} can also be described by a cut-paste construction: let Y_k be X_k with c_k and d_k removed, and let $Y(g)$ be a copy of Y for $g \in \Gamma$. Then

$$X_{k+1} = \left(\bigcup_{g \in \Gamma} Y_k(g) \right) \cup \{\text{1-cells } c_k(g), \, d_k(g)\}_{g \in \Gamma}$$

where $c_k(g)$ goes from (starting point of c_k) $\in Y_k(g) = Y$ to (endpoint of c_k) $\in Y_k(g + (1,0)) = Y_k$, and $d_k(g)$ goes from (starting point of d_k) $\in Y_k(g) = Y$ to (endpoint of d_k) $\in Y_k(g + (0,1)) = Y_k$. See Figure 3. Define $c_{k+1} = c_k(0,0)$ and $d_{k+1} = -c_k(1,1)$, i.e., $c_k(1,1)$ with reversed orientation, as illustrated in Figure 3.
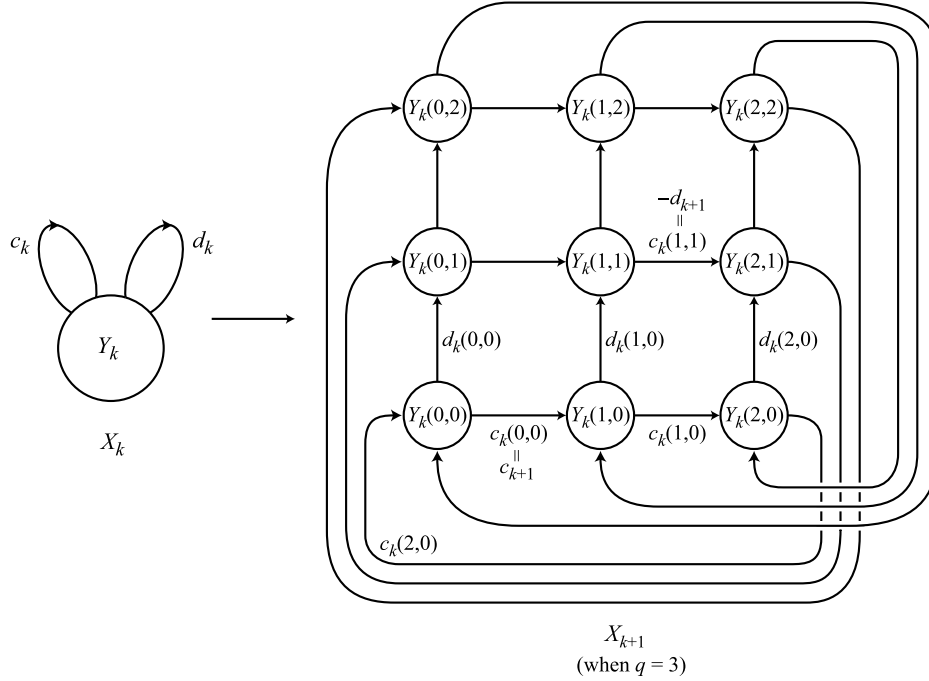
FIGURE 3. Construction of X_{k+1} from X_k

Let $\bar{X}_{k+1}$ be the 1-complex obtained from X_{n+1} by collapsing each $Y_n(g)$ ($g \in \Gamma$) to a point. For a path γ in X_{k+1}, we denote the image of γ in $\bar{X}_{k+1}$ by $\bar{\gamma}$. In particular the image of the 1-cells $c_k(g), d_k(g)$ are denoted by $\bar{c}_k(g), \bar{d}_k(g)$

We choose a basepoint $* \in X_k$ as follows: $* \in X_0 = S^1 \vee S^1$ is the wedge point, and $* \in X_{k+1}$ is defined to be the pre-image of $* \in X_k$ contained in $Y_k(0,0) \subset X_{k+1}$. Also, we sometimes regard $* \in X_k$ as a point in $Y_k(g)$.

Lemma 5.7. *For a 0-cell v in X_{k+1}, let γ_v and δ_v be the lift of α_{k+1} and β_{k+1} in X_{k+1} based at v, respectively. Note that $v \in Y_k(g) \subset X_{k+1}$ for some $g \in \Gamma$.*

(1) *If $v \neq * \in Y_k(g)$, then $\bar{\gamma}_v$ and $\bar{\delta}_v$ are null-homotopic (rel ∂) in $\bar{X}_{k+1}$.*
(2) *If $v = * \in Y_k(g)$, then $\bar{\gamma}_v$ and $\bar{\delta}_v$ are homotopic (rel ∂) to*

$$\bar{c}_k(g)\bar{d}_k(g+(1,0))\bar{c}_k(g+(0,1))^{-1}\bar{d}_k(g)^{-1} \text{ and}$$

$$\bar{c}_k(g)\bar{c}_k(g+(1,0))\bar{d}_k(g+(2,0))\bar{c}_k(g+(1,1))^{-1}\bar{d}_k(g+(1,0))^{-1}\bar{c}_k(g)^{-1}$$

in $\bar{X}_{k+1}$, respectively.

Proof. We use an induction on k. When $k = 1$, the conclusion is easily verified. Suppose it holds for k. Let $v' \in X_k$ be the image of v under $X_{k+1} \to X_k$. Let a and b be the lifts of α_k and β_k in X_k based at v'. Since $\alpha_{k+1} = \alpha_k\beta_k\alpha_k^{-1}\beta_k^{-1}$, γ_v is obtained by concatenating appropriate lifts of a, b, a^{-1}, and b^{-1} in X_{k+1}.

If $\bar{a}$ is homotopic (rel ∂) to a path which does not pass through $\bar{c}_k$ nor $\bar{d}_k$ in $\bar{X}_k$, then by the construction of X_{k+1} from X_k, any lift of a in X_{k+1} is contained in some $Y_k(-)$, and so $\bar{\gamma}$ is null-homotopic (rel ∂) in $\bar{X}_{k+1}$. The analogue for b holds too. From this it follows that $\bar{\gamma}_v$ can be non-null-homotopic only if any paths homotopic to $\bar{a}$ or $\bar{b}$ pass through either $\bar{c}_k$ or $\bar{d}_k$. By our conclusion for k, this can be satisfied only when $v' = * \in X_k$. Furthermore, if this is the case, it can be verified that the image in $\bar{X}_{k+1}$ of lifts of a, b in X_{k+1} are of the form $\bar{c}_n(-)$, $\bar{d}_n(-)$, respectively, so that $\bar{\gamma}_v$ is of the desired form. The conclusion for $\bar{\delta}_v$ is proved similarly. $\qquad\square$

Proof of Lemma 5.3. We will show that our α and $\{X_k\}$ satisfy the conclusion of Lemma 5.3. The remaining thing to show is that there is $f\colon \pi_1(X_n) \to \mathbb{Z}$ with the properties described in Lemma 5.3 (2). Choose a map $\bar{X}_n \to S^1$ which sends all 0-cells of $\bar{X}_n$ to a fixed basepoint in S^1, sends the 1-cells $\bar{c}_n = \bar{c}_n(0,0)$ and $\bar{c}_n(1,0)$ to a loop generating $\pi_1(S^1) = \mathbb{Z}$ and its inverse, respectively, and sends all other 1-cells to the basepoint. Let $f\colon \pi_1(X_n) \to \mathbb{Z}$ be the map induced by $X_n \to \bar{X}_n \to S^1$.

Then, by using Lemma 5.7, it is easily verified that f sends all the lifts of α to either -1, 0, or 1; for, if the image $\bar{\gamma}$ of a lift of α in $\bar{X}_n$ is not null-homotopic, then from Lemma 5.7 it follows that $\bar{\gamma}$ should be of the form described in Lemma 5.7 (2), which can pass through $\bar{c}_{n-1}(0,0)$ and $\bar{c}_{n-1}(1,0)$ at most once but cannot pass through both. Also, by Lemma 5.7 (2), (the image of) the lift of α based at $* \in X_n$ passes through $\bar{c}_{n-1}(0,0)$ exactly once but never passes through $\bar{c}_{n-1}(1,0)$. So, this lift is sent to 1 by f.

If γ is a loop in X_n which projects to a power of α_i in X, then it can be seen that $\bar{\gamma}$ in $\bar{X}_n$ is either null-homotopic or of the form $\left(\prod_i \bar{c}_n(g + (i,0))\right)^a$ or $\left(\prod_j \bar{d}_n(g + (0,j))\right)^a$. From the definition of f, it follows that f sends γ to 0. This shows the local triviality claim (see Remark 4.1). $\qquad\square$

6. Independence of links

Recall that a connected sum of two links L_1 and L_2 is defined by choosing disk basings; given a disk basing of each L_i, we obtain a string link β_i whose closure is L_i, and the closure of $\beta_1\beta_2$ is defined to be a connected sum of L_1 and L_2. In this section we investigate "independence" of links under connected sum.

For this purpose, we need a result due to Habegger and Lin [13]. Following [12, 13], we define a left action $\Sigma\cdot\colon \beta \to \Sigma\beta$ of a $2m$-string link Σ on an m-string link β as in Figure 4. The right action $\cdot\Sigma\colon \beta \to \beta\Sigma$ is defined similarly. Denote $\mathcal{S} = \{\Sigma \mid 1_m\Sigma = 1_m\}$ where 1_m denotes the trivial m-string link.
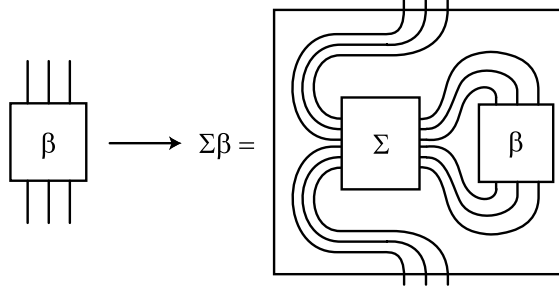

FIGURE 4. Left action of Σ on β

Proposition 6.1 ([13]). *The closures of two m-string links β_1 and β_2 are concordant (as links) if and only if β_2 is concordant to $\Sigma\beta_1$ (as string links) for some $2m$-string link $\Sigma \in \mathcal{S}$.*

Throughout this section, we assume $p = 2$. Fix $m > 1$ and n, and consider the string links $\beta(K_i)$ constructed in Section 5 (using $p = 2$). Namely, K_i is as in Lemma 5.2 and $\beta(K_i)$ is obtained from the trivial string link by infection using K_i along a curve realizing the homotopy class of the loop given by Lemma 5.3. Let L_i be the closure of $\beta(K_i)$.

Theorem 6.2. *Suppose that a connected sum of the a_iL_i and some local knots is $\mathbb{Z}_{(p)}$-coefficient $(n.5)$-solvable for some integers a_i, some disk basings, and some order of the a_iL_i. Then $a_i = 0$ for all i.*

As mentioned in Section 5, each $\beta(K_i)$ is (integrally) (n)-solvable and of vanishing Harvey's ρ_n-invariant, and thus so is L_i. Therefore, from Theorem 6.2, Theorem 1.4 follows immediately: the links L_i are (n)-solvable, has vanishing ρ_n-invariants, and independent over $\mathbb{Z}$ modulo $\mathcal{F}_{(n.5)}^{\mathrm{L}}$ and local knots.

Proof. We may assume that $a_i \geq 0$ by replacing L_i by $-L_i$ (and K_i by $-K_i$) if necessary. First we consider the special case that there is no local knot summand; suppose β'_{ij} $(1 \leq j \leq a_i)$ is a string link with closure L_i and the product β' of the β'_{ij} (in some order) is $\mathbb{Z}_{(p)}$-coefficient $(n.5)$-solvable. By Proposition 6.1, there is $\Sigma_{ij} \in \mathcal{S}$ such that β'_{ij} is concordant to $\Sigma_{ij}\beta(K_i)$. Let $\beta_{ij} = \Sigma_{ij}\beta(K_i)$. Then the product

$$\beta = \prod_{i,j} \beta_{ij}$$

is $\mathbb{Z}_{(p)}$-coefficient $(n.5)$-solvable (for some order of the factors) since β is concordant to β'.

Suppose not all a_i are zero. Choose the minimal i_0 such that $a_{i_0} \neq 0$. We will derive a contradiction by showing that a_{i_0} should be zero. For this purpose, we need the following facts: let $\{X_k\}$ be the p-tower constructed in Lemma 5.3. Then

(1) For each i, there is a character $\theta_i \colon \pi_1(X_n) \to \mathbb{Z}_{d_i}$ such that the p-structure $\mathcal{T}_i = (\{X_k\}, \theta_i)$ is locally trivial and satisfies $\lambda_{\mathcal{T}_i}(\beta(K_i)) = c \cdot \sigma_{K_i}(\zeta_{d_i})$ for some constant $c > 0$.

(2) For any p-structure $\mathcal{T} = (\{X_k\}, \theta)$ of height n with $\theta \colon \pi_1(X_n) \to \mathbb{Z}_d$, we have $\lambda_{\mathcal{T}}(\beta(K)) = \sum_k \sigma_K(\zeta_d^{s_k})$ for some $\{s_k\}$.

(1) was proved in the proof of Theorem 5.4 using Lemma 5.3. Observing that our infection curve α producing $\beta(K_i)$ lifts to the nth term of the p-tower, (2) follows immediately from Proposition 4.3, as in the proof of Theorem 5.4.

Among the factors $\beta_{i_0 j}$ $(1 \leq j \leq a_{i_0})$, choose an arbitrary one, say $\beta_{i_0 j_0}$. Appealing to (1) above, choose a p-structure $\mathcal{T}_{i_0 j_0}$ of height n such that $\lambda_{\mathcal{T}_{i_0 j_0}}(\beta(K_{i_0})) = c \cdot \sigma_{K_{i_0}}(\zeta_{d_{i_0}})$.

Let μ_{ij} be the composition of the preferred meridian map $X \to E_{\beta(K_i)}$ for $\beta(K_i)$ and the natural inclusion

$$E_{\beta(K_i)} \longrightarrow E_{\beta_{ij}} \longrightarrow E_\beta \longrightarrow M_\beta$$

and $\mu \colon X \to M_\beta$ be the preferred meridian map for β. Then, since β is an $\widehat{F}$-string link, μ and the μ_{ij} are p-tower maps by [2, Proposition 6.3]. So the p-structure $\mathcal{T}_{i_0 j_0}$ determines a p-structure $\mathcal{T}'$ for M_β via $\mu_{i_0 j_0}$, and then $\mathcal{T}'$ induces a p-structure $\mathcal{T}_{ij}$ for X via μ_{ij} for each i, j. Also, $\mathcal{T}'$ induces a p-structure $\mathcal{T}$ for X via μ.

From our choice of the p-structures, it follows that

$$\lambda_{\mathcal{T}}(\beta) = \sum_{i,j} \lambda_{\mathcal{T}}(\beta_{ij}) = \sum_{i,j} \lambda_{\mathcal{T}_{ij}}(\beta(K_i)).$$

By the property stated in Lemma 5.2 (1), we have

$$\lambda_{\mathcal{T}_{i_0 j_0}}(\beta(K_{i_0})) = c\sigma_{K_{i_0}}(\zeta_{d_{i_0}}) \neq 0.$$

For $i = i_0$ and $j \neq j_0$, we have

$$\lambda_{\mathcal{T}_{i_0 j}}(\beta(K_{i_0})) = \sum_k \sigma_{K_{i_0}}(\zeta_{d_{i_0}}^{s_k})$$

for some $\{s_k\}$ by (2) above. By the property stated in Lemma 5.2 (1), each summand of $\lambda_{\mathcal{T}_{i_0 j}}(\beta(K_{i_0}))$ is either zero or of the same sign with $\lambda_{\mathcal{T}_{i_0 j_0}}(\beta(K_{i_0}))$ (Here we need the assumption that $p = 2$). For $i < i_0$, there is no (i, j)-summand in the

above expression of $\lambda_{\mathcal{T}}(\beta)$ since $a_i = 0$ by our choice of i_0. For $i > i_0$, for some $\{s_k\}$ we have

$$\lambda_{\mathcal{T}_{ij}}(\beta(K_i)) = \sum_k \sigma_{K_i}(\zeta_{d_{i_0}}^{s_k}) = 0$$

by the property stated in Lemma 5.2 (2).

So it follows that $\lambda_{\mathcal{T}}(\beta) \neq 0$. This contradicts that β is $\mathbb{Z}_{(p)}$-coefficient $(n.5)$-solvable, by Theorem 1.2. It completes the proof when there is no local knot summand.

For the general case, suppose that the product of β and some local knots, say β_k, is $\mathbb{Z}_{(p)}$-coefficient $(n.5)$-solvable. In this case, we need an additional argument as described below. Observe that now we have

$$0 = \sum_{i,j} \lambda_{\mathcal{T}_{ij}}(\beta(K_i)) + \sum_k \lambda_{\mathcal{T}}(\beta_k).$$

So it suffices to show that $\lambda_{\mathcal{T}}(\beta)$ vanishes for any local knot β. Let denote $\mathcal{T}' = (\{M_k\}, \phi)$. Since $\mathcal{T}_{i_0 j_0} = (\{X_k\}, \theta_{i_0})$ is locally trivial and induced by $\mathcal{T}'$ via the meridian map $\mu_{i_0 j_0}$, any loop in M_n that projects to a power of a meridian in M_β is in the kernel of ϕ, by Lemma 4.4 (see also Remark 4.1). Applying Lemma 4.4 to the preferred meridian map μ, it follows that $\mathcal{T}$ is locally trivial. Therefore $\lambda_{\mathcal{T}}(\beta) = 0$ for any local knot β, by Theorem 4.2. This finishes the proof.   □

The arguments in [2, Section 7] (in particular see Lemma 7.7 of [2]) prove that the nth iterated Bing double $BD_n(K)$ of a knot K is the closure of a string link obtained from the trivial string link by infection along a curve α, which satisfies the conclusion of our Lemma 5.3; while the local triviality condition is not mentioned in [2], it can be satisfied by a minor change of the construction of the character in [2]. So, our argument shows the following:

Theorem 6.3. *Suppose $\{K_i\}$ is a family of knots satisfying Lemma 5.2. Then, for any n, the links $BD_n(K_i)$ are (n)-solvable, have vanishing ρ_n-invariant, and are independent over $\mathbb{Z}$ modulo $\mathcal{F}^{\mathrm{L}}_{(n.5)}$ and local knots.*

Using a similar technique, we can also show the independence of certain 2-torsion iterated Bing doubles considered in [2], namely we can show Theorem 1.5: there are infinitely many amphichiral knots K_i such that for any n, the links $BD_n(K_i)$ are 2-torsion, (n)-solvable, and independent over $\mathbb{Z}_2$ modulo $\mathcal{F}^{\mathrm{L}}_{(n+1.5)}$ and local knots.

Proof of Theorem 1.5. In this proof we use tools from algebraic number theory: the discriminant map

$$\mathrm{dis} \colon L^0(\mathbb{Q}(\zeta_d)) \longrightarrow \frac{\mathbb{Q}(\zeta_d + \zeta_d^{-1})^\times}{\{z\bar{z} \mid z \in \mathbb{Q}(\zeta_d)^\times\}}$$

and the norm residue symbol $(x, y)_q \in \{\pm 1\}$ which is defined for $x, y \in \mathbb{Q}(\zeta_d + \zeta_d^{-1})^\times$ and for a prime q of $\mathbb{Q}(\zeta_d + \zeta_d^{-1})$. (Here $\{\pm 1\}$ is regarded as a multiplicative group of order two.) Essentially what we need is the following fact: dis is a group homomorphism, and when $d = 4$ (i.e., $\zeta_d = \sqrt{-1}$), $(x, -1)_q = 1$ for all prime q if $x \in \mathbb{Q}^\times$ is of the form $z\bar{z}$ for some $z \in \mathbb{Q}(\sqrt{-1})^\times$. Interested readers who are not familiar with these tools are referred to Section 4.5 of [2] and Section 3.4 of [3], which provide more detailed accounts on the discriminant and norm residue symbols for non-experts of algebraic number theory.

We will show that the family $\{K_i\}$ of knots constructed in the proof of Corollary 8.5 (1) in [2] satisfies our conclusion. It was shown that $BD_n(K_i)$ is 2-torsion and (n)-solvable in [2]. The properties stated below, which are shown by the arguments of the proof of Corollary 8.5 of [2] (see also Proposition 5.6 of [2]), are the only facts on the K_i that we need in order to prove the independence: $BD_n(K_i)$

is the closure of a string link β_i which admits a "dual prime" p_i satisfying the following:

(1) For each i, there is a locally trivial $\mathbb{Z}_4$-valued 2-structure $\mathcal{T}_i$ of height $n+1$ such that
$$\left(\operatorname{dis}\lambda_{\mathcal{T}_i}(\beta_i), -1\right)_{p_i} = -1.$$

(2) If $i \neq j$, then for any $\mathbb{Z}_4$-valued 2-structure $\mathcal{T}$ of height $n+1$,
$$\left(\operatorname{dis}\lambda_{\mathcal{T}}(\beta_j), -1\right)_{p_i} = +1.$$

(Note that $\lambda_{\mathcal{T}}(-)$ and $\lambda_{\mathcal{T}_i}(-)$ are always in $L^0(\mathbb{Q}(\sqrt{-1}))$ so that the discriminant is in $\mathbb{Q}$, since the p-structures are $\mathbb{Z}_4$-valued.)

Now suppose that for some finite nonempty subset I of $\mathbb{N}$, a connected sum of $\{L_i\}_{i \in I}$ and some local knots is $\mathbb{Z}_{(p)}$-coefficient $(n+1.5)$-solvable. Choose any $i_0 \in I$. Then, by appealing to Habegger-Lin's Proposition 6.1 and by choosing appropriate locally trivial p-structures via meridian maps as in the proof of Theorem 6.2, we can see that there are p-structures $\mathcal{T}_i$ such that
$$0 = \sum_{i \in I} \lambda_{\mathcal{T}_i}(\beta_i)$$

where $\mathcal{T}_{i_0}$ satisfies $(\operatorname{dis}(\lambda_{T_{i_0}}(\beta_{i_0}), -1)_{p_{i_0}} = -1$ as in (1). Taking the discriminant and evaluating the norm residue symbol $(\,\cdot\,, -1)_{p_{i_0}}$, we have
$$1 = \prod_{i \in I} \left(\operatorname{dis}(\lambda_{T_i}(\beta_i), -1)_{p_{i_0}}\right).$$

But, by (2) we have $(\operatorname{dis}(\lambda_{T_i}(\beta_i), -1)_{p_{i_0}} = 1$ for $i \neq i_0$. This is a contradiction. $\square$

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Department of Mathematics, Pohang University of Science and Technology, Pohang, Kyungbuk 790–784, Republic of Korea

E-mail address: jccha@postech.ac.kr